Journal of Computational Physics 257 (2014) 757–781

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A perfectly matched layer for the time-dependent wave equation in heterogeneous and layered media

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ARTICLE INFO

Article history:
Received 19 April 2013
Received in revised form 9 October 2013
Accepted 11 October 2013
Available online 18 October 2013

Keywords:
Interface waves
Guided waves
Reflected waves
Refracted waves
Surface waves
Perfectly matched layer
High order accuracy
Stability
SBP–SAT

ABSTRACT

A mathematical analysis of the perfectly matched layer (PML) for the time-dependent wave equation in heterogeneous and layered media is presented. We prove the stability of the PML for discontinuous media with piecewise constant coefficients, and derive energy estimates for discontinuous media with piecewise smooth coefficients. We consider a computational setup consisting of smaller structured subdomains that are discretized using high order accurate finite difference operators for approximating spatial derivatives. The subdomains are then patched together into a global domain by a weak enforcement of interface conditions using penalties. In order to ensure the stability of the discrete PML, it is necessary to transform the interface conditions to include the auxiliary variables. In the discrete setting, the transformed interface conditions are crucial in deriving discrete energy estimates analogous to the continuous energy estimates, thus proving stability and convergence of the numerical method. Finally, we present numerical experiments demonstrating the stability of the PML in a layered medium and high order accuracy of the proposed interface conditions.

1. Introduction

High order accurate schemes have been demonstrated to be more efficient than lower (first and second) order accurate methods, for wave propagation problems in smooth domains [20,24]. In order for high order methods to succeed for wave propagation problems in unbounded domains, artificial boundaries introduced to limit the computational domain must be closed with reliable and accurate boundary conditions such that waves traveling out of the domain disappear without reflections. Otherwise, waves traveling out of the domain generate spurious reflections at the boundaries which will travel into the domain and pollute the solutions. The perfectly matched layer (PML) [7,8,5,23,13,3] has emerged as an effective tool to simulate the absorption of waves in numerical wave simulators and can be used as an alternative to high order non-reflecting boundary conditions (NRBC) [17,18,28,21,22,19]. An NRBC, on the one hand, is defined on an artificial boundary such that little or no spurious reflections occur as a wave passes the boundary. The PML, on the other hand, is constructed by extending the domain to a layer of finite thickness where the underlying equations are transformed such that waves traveling into the layer are absorbed without reflections. That said, when the PML is discretized, the discrete PML can allow some numerical reflections. However, for a well designed numerical method artificial reflections introduced by numerical approximations should converge to zero as the mesh is refined.

PMLs and NRBCs, with some exceptions [15,14,28,22], are often constructed and analyzed by assuming an infinite domain with constant coefficients and damping. However, real media are heterogeneous and discontinuous. For example in underwater acoustics it is relevant to consider waveguides consisting of several layers such as air, water, soft and hard sediment and bedrock layers. Applications arising in geophysics and electromagnetic problems can be composed of layers of rock, water and possibly oil. The main motivation of this paper is to rigorously establish the practicability and reliability of the PML in realistic media for problems arising in oil exploration, earthquake engineering, underwater acoustics, oceanography,

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.10.022

ultrasonics and ground penetrating radar technologies. We will prove the well-posedness and stability of the PML for the wave equation in heterogeneous and layered media. We will develop a provably stable high order accurate finite difference approximation of the PML in layered media. For the continuous problem, the term stability refers to asymptotic stability. That is, the continuous problem is stable if all solutions remain bounded for all time. In the discrete setting, stability denotes standard numerical stability. A numerical method is stable if at a fixed time interval the numerical errors remain bounded as the grid is refined.

For time dependent problems, it is very important that the PML is stable. Any growth in the PML can propagate into the computational domain and pollute the solution everywhere. For constant coefficients problems in unbounded domains, stability can be investigated using standard Fourier methods, see [13,3,5]. In bounded and semi-bounded domains the stability of the PML has also been analyzed [15,14] for certain classes of boundary conditions using normal mode analysis. To the best of our knowledge, the present study is the first where a mathematical analysis of the PML in discontinuous or heterogeneous media is presented. There are cases in the literature though where the PML has been reported to support growing solutions when material parameters are discontinuous [2]. In this paper we analyze the stability of the PML for the wave equation in vertically stratified media. Firstly, we use modal analysis, and secondly, we derive energy estimates.

As a model we consider the scalar wave equation in layered media with piecewise smooth material properties. The PML is derived by a complex change of spatial variables in the Laplace space. To localize the PML in time we chose auxiliary variables and invert the Laplace transforms. In order to ensure theoretical perfect matching we have assumed that in the far field, where the PML is placed, there are no sources, and the material properties are invariant in the direction normal to the PML. Note that the material parameters can vary, even discontinuously, in the tangential directions. The primary objective of this paper is to establish the well-posedness and stability of the continuous PML in layered and heterogeneous media. The first main result in this paper is a proof of stability using a modal analysis for the PML in layered media with piecewise constant material properties. Proving energy estimates for a PML is not trivial, there are only very few such estimates, see for examples [3,14,23]. The second main result in the paper is the derivation of an energy estimate for the PML in layered media with piecewise smooth material properties and variable damping. The energy estimate establishes the well-posedness of the PML in heterogeneous media.

In theory, the PML is a continuous unbounded layer surrounding the artificial boundary of a computational domain. However, when numerical methods are used, the PML changes to a discrete, finite width layer. In the discrete setting, in order to ensure accuracy and discrete stability, extra care should be taken when using the PML. For instance, if there are physical boundaries extending into the PML, the underlying boundary conditions must also be extended from the physical domain into the PML [15,14,30,31]. The second objective of this paper is to derive a high order accurate, provably stable and convergent numerical method for the PML in layered media. Firstly, we derive an equivalent set of transformed interface conditions giving identical results for the continuous problem, but different results numerically. To approximate spatial derivatives, we use high order finite difference schemes satisfying the summation-by-parts (SBP) rule [27,26]. The interface conditions are enforced weakly using a penalty technique referred to as the simultaneous approximation term (SAT) method [11,27]. In the absence of the PML our interface treatment is similar to the method developed in [27]. However, the penalty parameters derived in this paper are sharp and the technique in which they are derived are different from those of [27]. Another important result of this paper is that in the discrete setting, the transformed interface conditions are crucial in the derivation of a discrete energy estimate analogous to the continuous energy estimate. Furthermore, we prove convergence of the numerical method. We present numerical simulations verifying high order accuracy and the stability of the PML in layered media.

The outline of the paper is as follows. In Section 2, we introduce the model problem and derive the PML. The continuous analysis and energy estimates of the PML are presented in Section 3. In Section 4, we discretize the PML, derive discrete energy estimates and prove convergence of the numerical method. Numerical simulations and discussions of the results are presented in Section 5. We summarize the paper and suggest future work in the last section.

2. The wave equation and the perfectly matched layer

In this section we present the wave equation and the PML in heterogeneous and layered media.

2.1. The wave equation

Consider the scalar wave equation in a two space dimensional heterogeneous lossless unbounded media,

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot (\mu \nabla u) + f(x, y, t), \quad -\infty \leqslant x, y \leqslant \infty,\ t > 0, \tag{1}$$

with the initial data

$$u(x, y, 0) = u_0(x, y), \qquad \frac{\partial u}{\partial t}(x, y, 0) = v_0(x, y). \tag{2}$$

Here $u(x, y, t)$ is the wave field, $f(x, y, t)$ is an internal forcing, t denotes time, x, y are the spatial variables, ∇ is the gradient operator, $\rho(x, y) > 0$, $\mu(x, y) > 0$ are the material parameters characterizing the media. We define $c(x, y) =$