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Well-Balanced Adaptive Mesh Refinement for shallow water flows ☆

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ABSTRACT

Well-balanced shock capturing (WBSC) schemes constitute nowadays the state of the art in the numerical simulation of shallow water flows. They allow to accurately represent discontinuous behavior, known to occur due to the non-linear hyperbolic nature of the shallow water system, and, at the same time, numerically maintain stationary solutions. In situations of practical interest, these schemes often need to be combined with some kind of adaptivity, in order to speed up computing times. In this paper we discuss what ingredients need to be modified in a block-structured AMR technique in order to ensure that, when combined with a WBSC scheme, the so-called 'water at rest' stationary solutions are exactly preserved.

1. Introduction

The shallow water equations are a non-linear, hyperbolic, system of balance laws, which are obtained from the Navier–Stokes equations by depth averaging, after neglecting effects such as turbulence or shear stress. If only the effect of the bottom elevation, or bathymetry is considered, they take the following form

$$\begin{aligned} &h_t + \operatorname{div}(hv) = 0 \\ &(hv)_t + \operatorname{div}\left(hv \otimes v + \frac{gh^2}{2} I_2\right) = -gh\nabla z, \end{aligned} \tag{1}$$

where h denotes water depth, $v = (v^x, v^y)$ is the depth-averaged velocity, g is the gravity acceleration, z is the bottom elevation and I_2 is the 2×2 identity matrix. We will also denote the discharge by $q := hv$. This system is widely used in many applications to model flows in river and coastal areas, and has received a lot of attention in the scientific community during the last ten to fifteen years. There has been a tremendous research effort towards the development of numerical techniques for the shallow water equations. This effort is due in part to the many modeling applications of shallow water flows, but it is also due to the fact that there are specific difficulties in the numerical simulation of this system that make the problem attractive and challenging.

On a flat bathymetry, the shallow water equations (1) become a homogeneous system of conservation laws. Their solutions may develop discontinuities, even when the initial flow is smooth, which requires the use of shock-capturing schemes in order to ensure a proper handling of discontinuities in numerical simulations concerning this system of equations.

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The presence of a non-flat bathymetry leads to the inclusion of source terms in the system related to the bottom geometry. It is well known that naive discretizations of these source terms may lead to spurious, numerical, waves that can obscure, or even ruin, the real solution that needs to be computed. This spurious numerical behavior occurs when computing stationary, or nearly stationary, solutions, for which the balance between the convective fluxes and the source terms associated to the bathymetry is not respected by the numerical scheme. Well-balanced schemes [8,20] are specifically designed in order to maintain this balance, to machine accuracy if possible, and Well-balanced Shock Capturing (WBSC) schemes constitute nowadays the state of the art in the numerical simulation of shallow water flows.

Robust and accurate WBSC schemes often have a high computational cost, which is related to the fact that they incorporate upwinding through characteristic information that needs to be computed at each cell boundary in the computational domain, high order reconstruction procedures, and a sophisticated numerical treatment of the bathymetry source term. In situations of practical interest, it is highly desirable to combine a WBSC scheme with an adaptive technique that can lower its high computational cost in 2D simulations [18,6,23,27,24].

The efficiency of an AMR algorithm is related to the reliability of the mesh adaption procedure, which is usually controlled by user-dependent thresholding parameters. Good efficiency factors are obtained when the thresholding parameter is relatively large, however, the use of an 'efficient' thresholding parameter might lead to spurious numerical behavior, akin to that observed when a non-Well-Balanced (NWB) numerical scheme is used on a uniform mesh, when computing stationary or nearly stationary solutions to the water shallow model, even if the underlying solver is a WBSC scheme.

In this work we analyze a block structured AMR technique developed in [4] and point out that, in addition to using a WBSC scheme as the underlying scheme in the AMR process, it is necessary to implement Well-Balanced interpolatory techniques in the transfer operators involved in the multi-level grid structure in order for the combined AMR-WBSC scheme to maintain its well-balanced character.

The paper is organized as follows: In Section 2 we briefly recall the underlying WBSC scheme used by the block structured AMR technique and in Section 3 we recall the main ingredients of this technique, identifying those which are potentially responsible for the Well-Balance (WB) loss. In Section 4 we describe the necessary corrections to obtain a WB-AMR code and in Section 5 we show several numerical experiments that support our discussion. We close in 6 with some conclusions and perspectives for future work.

2. Well-balanced schemes for shallow water flows

The shallow water system (1) admits stationary solutions, in which non-zero flux-gradients are exactly balanced by the source terms. Such solutions, along with their perturbations, are difficult to capture numerically because straightforward discretizations of the source term fail to preserve this balance. Computing these solutions is indeed a challenge and there is a large body of recent research concerning numerical techniques that incorporate the necessary balance in their discrete design (e.g. [28,9,18,16,33,31,12]). Such schemes are termed well-balanced (WB) schemes after the work of Leroux and collaborators [20,21]. Bermúdez and Vázquez-Cendón, in an independent work [8], introduced the concept of the C*-property*. A scheme is said to satisfy the *exact* C-property if it preserves exactly the 'water at rest' stationary solution. Schemes that satisfy the exact C-property are WB for quiescent steady states.

All WBSC schemes preserve exactly the 'water at rest' stationary solution, for which $v^x = v^y = 0$ and $h + z = C$ (constant). However, as we shall see later on, the 'water at rest' might not be exactly preserved if the same scheme is used in a multi-scale framework. Our goal in this paper is to address the issue of *well-balancing* when a WBSC scheme is used as the underlying solver within a block-structured AMR technique.

The numerical experiments in this paper are carried out using a WBSC scheme developed in [28,14], which preserves exactly the water at rest steady state. For the sake of completeness, we give next a brief description of the scheme for the simpler 1D shallow water model, which takes the form ($v = v^x$)

$$
\begin{aligned}
&h_t + (hv)_x = 0\\
&(hv)_t + \left(hv^2 + \frac{gh^2}{2}\right)_x = -ghz_x.
\end{aligned} \tag{2}
$$

Using the notation:

$$
u = [h \quad hv]^T, \qquad f(u) = \left[hv \quad hv^2 + \tfrac{gh^2}{2}\right]^T, \qquad s(x,u) = [0 \quad -ghz_x]^T,
$$

system (2) can be written as:

$$
u_t + f(u)_x = s(x,u)
$$

which, in turn, can be rewritten in the *homogeneous* form: $u_t + g[u]_x = 0$, where the functional g (dependent on f and s) acts on $u = u(x,t)$ as: