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Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics

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ABSTRACT

We present hybridizable discontinuous Galerkin methods for solving steady and time-dependent partial differential equations (PDEs) in continuum mechanics. The essential ingredients are a local Galerkin projection of the underlying PDEs at the element level onto spaces of polynomials of degree k to parametrize the numerical solution in terms of the numerical trace; a judicious choice of the numerical flux to provide stability and consistency; and a global jump condition that enforces the continuity of the numerical flux to arrive at a global weak formulation in terms of the numerical trace. The HDG methods are fully implicit, high-order accurate and endowed with several unique features which distinguish themselves from other discontinuous Galerkin methods. First, they reduce the globally coupled unknowns to the approximate trace of the solution on element boundaries, thereby leading to a significant reduction in the degrees of freedom. Second, they provide, for smooth viscous-dominated problems, approximations of all the variables which converge with the optimal order of $k+1$ in the L^2-norm. Third, they possess some superconvergence properties that allow us to define inexpensive element-by-element postprocessing procedures to compute a new approximate solution which may converge with higher order than the original solution. And fourth, they allow for a novel and systematic way for imposing boundary conditions for the total stress, viscous stress, vorticity and pressure which are not naturally associated with the weak formulation of the methods. In addition, they possess other interesting properties for specific problems. Their approximate solution can be postprocessed to yield an exactly divergence-free and $\boldsymbol{H}(\mathrm{div})$-conforming velocity field for incompressible flows. They do not exhibit volumetric locking for nearly incompressible solids. We provide extensive numerical results to illustrate their distinct characteristics and compare their performance with that of continuous Galerkin methods.

1. Introduction

Partial differential equations (PDEs) in continuum mechanics have attracted considerable interest from physicists and mathematicians over centuries. This interest stems from the curiosity of real-world phenomena and the attempt to model them by using physical balance laws and mathematical equations. Except for very few cases in which they admit analytical solutions, PDEs are numerically solved in most physical problems of interest. The development of numerical methods for PDEs and their applications have given birth to two new emerging fields in engineering and science, namely, computational mechanics and computational mathematics. Indeed, numerical methods for PDE have played and will continue to play an important role in aiding our understanding of complex physical phenomena without the need of performing experiments.

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They have also helped and will continue to help us to design airplanes, cars, transportation systems, telecommunication systems, biomedical devices, among other important components or systems.

Over the past decades, the finite element method has been among the most popular techniques for numerically solving PDEs. There are several discretization strategies within the finite element method. They include continuous Galerkin methods, SUPG methods [31], least-squares finite element methods [32], spectral element methods [52], mixed finite element methods [8], extended finite element methods [40], discontinuous Galerkin methods [58], discontinuous Petrov–Galerkin methods [20], and other finite element methods. Each method has its own strengths and weaknesses that make it ideal for some applications, but not the best choice for others. As a result, finite element practitioners typically make a choice of methods based on the particular problem they want to solve. For instance, discontinuous Galerkin methods mainly find their domain of applications in computational fluid dynamics, while continuous Galerkin methods are widely used to solve problems in solid and fluid mechanics.

In recent years, considerable attention has been turned to discontinuous Galerkin (DG) methods for the numerical solution of PDEs [3,5,6,18,19,23,28,29,33,38,41,54,55,58,61]. DG methods possess several attractive features. In particular, they are flexible for complicated geometry, locally conservative, high-order accurate, highly parallelizable, low dissipative, and more stable than continuous Galerkin (CG) methods for convection-dominated problems. However, in spite of all these advantages, DG methods have not yet made a more significant impact for practical applications as we would hope. This is largely due to the main criticism that DG methods are computationally extensive since they have too many degrees of freedom due to nodal duplication at the element boundary interfaces. More specifically, assuming about six linear tetrahedral elements per node, the number of unknowns in a DG system would approximately 24 times the number of unknowns in the corresponding CG system. For cubic tetrahedral elements, the degrees-of-freedom ratio between DG and CG reduces to approximately 5. The high computational cost and memory storage are a major impediment to the widespread application of DG methods for real-world problems. Therefore, it would be highly desirable to develop new DG methods that have all the advantages of DG methods (analyzed in [3]), and yet are competitive with continuous Galerkin methods.

Recently, researchers in computational mechanics and mathematics have developed more efficient DG methods—the multiscale discontinuous Galerkin (MDG) method [30] and the embedded discontinuous Galerkin (EDG) method [25]—as a timely response to the above criticism of DG methods. Further development of the MDG method for the incompressible Navier–Stokes equations leads to the so-called Galerkin interface stabilisation (GIS) method [35]. The EDG methods for the compressible Euler and Navier–Stokes equations have been proposed in [53]. Both the MDG and EDG methods are devised to solve for the numerical trace of the solution as globally coupled unknowns. Therefore, the matrix system of these DG methods is similar to that of the statically condensed version of CG methods. However, both the MDG and EDG methods are not locally conservative and have the same convergence rates as CG methods.

More recently, a newer class of DG methods—the so-called hybridizable discontinuous Galerkin (HDG) method—is first introduced in [13] for elliptic problems. The HDG method for elliptic problems is then analyzed in [10,15,16] where it is shown that the HDG method has many common features with the Raviart–Thosmas (RT) mixed method [57] and the Brezzi–Douglas–Marini (BDM) mixed method [7]. In particular, the HDG method provides optimal convergence rates of order $k+1$ for all the approximate variables. Furthermore, element-by-element postprocessing is developed to increase the convergence rate of the approximate scalar variable to $k+2$ for $k \geqslant 1$. Shortly later, HDG methods have been further developed and analyzed for linear convection–diffusion problems [47], nonlinear convection–diffusion problems [11,48,65], Stokes problems [12,14,17,49], and incompressible Navier–Stokes equations [45,46,50]. A unique feature of these HDG methods is that the approximate velocity, pressure and velocity gradient converge with the optimal order $k+1$ in the L^2-norm for smooth diffusion-dominated incompressible flows. In such case, a local postprocessing scheme proposed in [14,45] can be used to obtain an exactly divergence-free, $\boldsymbol{H}(\mathrm{div})$-conforming velocity which converges with order $k+2$ for $k \geqslant 1$.

Although computational fluid dynamics has been their main domain of applications, DG methods have recently been introduced to computational solid mechanics [4,26,27,51,64]. However, most current DG methods are far from competitive with CG methods for solving problems in solid mechanics because they provide the same accuracy often at a significantly greater cost than CG methods. After all, most of problems in solid mechanics are not convection-dominated, as such the efficiency and applicability of many existing DG methods are limited to quite few problems. However, owing to their excellent convergence properties for diffusion-dominated problems and reduced degrees of freedom, HDG methods can be competitive with CG methods and thus can be relevant for solving a large number of problems in solid mechanics. In particular, HDG methods have been introduced for solving linear elasticity and shell problems [62] as well as linear acoustic and elastic wave equations [44]. It has been demonstrated for linear acoustic and elastic wave equations that HDG methods produce significantly more accurate solution than the standard CG method even for the same global degrees of freedom (see [44]).

In this paper, we further develop hybridizable discontinuous Galerkin methods for solving steady and time-dependent PDEs in continuum mechanics, with the objective of broadening the application of HDG methods in computational fluid dynamics and computational solid mechanics. The essential ingredients are a local Galerkin projection of the underlying PDEs at the element level onto spaces of polynomials of degree k to parametrize the numerical solution in terms of the numerical trace; a judicious choice of the numerical flux to provide stability and consistency; and a global jump condition that enforces the continuity of the numerical flux to arrive at a global weak formulation in terms of the numerical trace. The HDG methods are fully implicit, high-order accurate and endowed with several unique features which distinguish themselves from other discontinuous Galerkin methods. First, it reduces the globally coupled unknowns to the approximate trace of the solution on element boundaries, thereby leading to a significant reduction in the degrees of freedom. Second, it