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Boundary element dynamical energy analysis: A versatile method for solving two or three dimensional wave problems in the high frequency limit

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ABSTRACT

Dynamical energy analysis was recently introduced as a new method for determining the distribution of mechanical and acoustic wave energy in complex built up structures. The technique interpolates between standard statistical energy analysis and full ray tracing, containing both of these methods as limiting cases. As such the applicability of the method is wide ranging and additionally includes the numerical modelling of problems in optics and more generally of linear wave problems in electromagnetics. In this work we consider a new approach to the method with enhanced versatility, enabling three-dimensional problems to be handled in a straightforward manner. The main challenge is the high dimensionality of the problem: we determine the wave energy density both as a function of the spatial coordinate and momentum (or direction) space. The momentum variables are expressed in separable (polar) coordinates facilitating the use of products of univariate basis expansions. However this is not the case for the spatial argument and so we propose to make use of automated mesh generating routines to both localise the approximation, allowing quadrature costs to be kept moderate, and give versatility in the code for different geometric configurations.

1. Introduction

Predicting the wave energy distribution of the vibro-acoustic response of a complex mechanical system is a challenging task, especially in the mid-to-high frequency regime. Standard numerical tools such as finite element methods become inefficient, and ray or thermodynamic approaches are often employed to model the wave energy flow through the structure. Popular methods are *statistical energy analysis* (SEA) [1–3], in which the mean energy flow between subsystems is assumed to be proportional to the energy gradient, and the *ray tracing technique*, in which the wave intensity distribution is determined by summing over contributions of a potentially large number of ray paths [4–6].

SEA is in fact a low resolution ray tracing method [7,8] leading to small numerical models compared to ray tracing. This efficiency saving comes at a price, however: SEA has no spatial resolution of the energy distribution within subsystems and becomes unreliable whenever long range correlations in the ray dynamics are present. The recently developed *dynamical energy analysis* (DEA) [8,9] provides a tool which interpolates between SEA and a full ray tracing analysis and can overcome some of the problems mentioned above at a relatively small computational overhead. DEA thus enhances the range of applicability of standard SEA and gives bounds on the range of applicability of SEA. Related methods have been discussed

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previously in the context of wave chaos [10] and structural dynamics [11]. In particular Langley's *wave intensity analysis* (WIA) [12,13] and Le Bot's thermodynamical high frequency boundary element method [14–16] include details of the underlying ray dynamics. The approach employed here differs from these methods by considering multiple reflections in terms of linear operators. Representing these operators in terms of basis function expansions then leads to SEA-type equations.

In this work we develop a new approach to DEA suitable for modelling three-dimensional problems. The present DEA methods rely on the fact that one can easily parametrise the boundary of the region being modelled, and then apply an orthonormal basis approximation over the resulting boundary phase space coordinate system. In two dimensions this is simple as the boundary may be parametrised along its arc-length and the associated momentum (or direction) coordinate taken tangential to the boundary. The basis can be any suitable (scaled) univariate basis in both position and momentum, such as a Fourier basis [8] or Chebyshev polynomials [9]. Defining a suitable parametrisation for the spatial coordinate in three-dimensions becomes much more difficult. In momentum space spherical polar coordinates may be employed and so these problems do not arise.

In order to develop a flexible code we employ automated mesh generating routines to provide a widely applicable parametrisation of the boundary surface for general three-dimensional structures via triangulation. The precision of the spatial approximation may then be improved by refining the mesh, avoiding the issue of finding a suitable basis. One avenue for potential future study stems from the fact that it is possible to define an orthogonal basis on a general triangle which reduces to Legendre polynomials along one edge of the domain triangle [17]. However, in this work we restrict to a piecewise constant approximation on each element of the mesh for reasons of both simplicity and to keep the associated quadrature costs moderate for the three dimensional case.

For the choice of momentum basis we may take a product univariate basis as mentioned above. It is preferable if this basis is orthogonal with respect to the standard L^2 inner product for consistency with both the piecewise constant spatial approximation, and the SEA limit when the lowest order momentum basis is applied and continuity is enforced across the mesh. The main choices are either a Fourier basis or Legendre polynomials. In this work we choose Legendre polynomials due to better convergence properties in the absence of periodic boundary conditions [18] and for consistency with the approach in [17] should we wish to include a spatial basis in future work.

The remainder of the paper is structured as follows. In Section 2, the ray tracing approximation is discussed and related to the Green function using short wavelength asymptotics. In Section 3, the concept of phase-space operators is introduced in order to represent the propagation of ray densities in terms of boundary integrals. The discretization of the method using spatial meshing procedures and basis function approximations in direction space is then detailed. Decomposition of the method for problems with multiple subsystems is then discussed along with links between the method and SEA. In Section 4 the application of boundary element DEA to two-dimensional examples is discussed and verified against previous work. Finally some three-dimensional examples are considered.

2. Wave equations and asymptotics

It is assumed that the system as a whole is characterized by a linear wave equation describing the overall wave dynamics including damping and radiation in a finite domain $\Omega \subset \mathbb{R}^d,\ d = 2$ or 3. In this work only stationary problems with continuous, monochromatic energy sources are considered. We split the system into N_Ω subsystems and consider the scalar wave equation for acoustic pressure waves in each homogeneous sub-domain Ω_i, with local wave velocity $c_i, i = 1, \ldots, N_\Omega$ and $\Omega = \bigcup_{i=1}^{N_\Omega} \Omega_i$. Extensions to more complicated systems with different wave operators in different parts of the system can be treated with the same techniques as long as the underlying wave equations are linear, see the discussion in Ref. [8].

The general problem of determining the response of a system to external forcing with angular frequency ω at a source point $r_0 \in \Omega_0$ can then be reduced to solving

$$(k_i^2 - \hat{H})G(r, r_0; \omega) = -F_0\delta(r - r_0), \quad i = 1, \ldots, N_\Omega, \tag{1}$$

with $\hat{H} = -\Delta$. The Green function G represents an acoustic pressure wave where F_0 is a unit amplitude forcing term with units $kg\ s^{-2}$. The solution point is denoted $r \in \Omega_i$ and δ is the Dirac delta distribution. Furthermore, $k_i = \omega/c_i + i\mu_i/2$ is a complex valued wavenumber, where the imaginary part represents a subsystem dependent damping coefficient μ_i. Throughout this work we take $i = \sqrt{-1}$ unless used as a subscript, in which case it is an index over the number of subsystems. The wave energy density induced by the source is then given as

$$\varepsilon(r, r_0; \omega) = \frac{|G(r, r_0; \omega)|^2}{\varrho_i c_i^2}, \tag{2}$$

for $r \in \Omega_i$ where ϱ_i is the density of the medium in Ω_i. The linear wave operator $\hat{H}$ can naturally be associated with the underlying ray dynamics via the Eikonal approximation; for a more detailed derivation, see Ref. [8,19,20]. Using small wavelength asymptotics, the Green function in Eq. (1) may be written as a sum over *all* classical rays from r_0 to r for fixed kinetic energy of the hypothetical ray particle. One obtains [20,21]

$$G(r, r_0; \omega) \approx \frac{\pi}{(2\pi i)^{(d+1)/2}} \sum_{j: r_0 \to r} A_j e^{i(k_i L_j - i\nu_j \pi/2)}, \tag{3}$$