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Numerical experiments on the efficiency of local grid refinement based on truncation error estimates

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ABSTRACT

Local grid refinement aims to optimise the relationship between accuracy of the results and number of grid nodes. In the context of the finite volume method no single local refinement criterion has been globally established as optimum for the selection of the control volumes to subdivide, since it is not easy to associate the discretisation error with an easily computable quantity in each control volume. Often the grid refinement criterion is based on an estimate of the truncation error in each control volume, because the truncation error is a natural measure of the discrepancy between the algebraic finite-volume equations and the original differential equations. However, it is not a straightforward task to associate the truncation error with the optimum grid density because of the complexity of the relationship between truncation and discretisation errors. In the present work several criteria based on a truncation error estimate are tested and compared on a regularised lid-driven cavity case at various Reynolds numbers. It is shown that criteria where the truncation error is weighted by the volume of the grid cells perform better than using just the truncation error as the criterion. Also it is observed that the efficiency of local refinement increases with the Reynolds number. The truncation error is estimated by restricting the solution to a coarser grid and applying the coarse grid discrete operator. The complication that high truncation error develops at grid level interfaces is also investigated and several treatments are tested.

1. Introduction

The finite volume method is a popular method in Computational Fluid Dynamics (CFD): The domain is divided into a number of Control Volumes (CVs) using a grid, the differential equations are integrated over each CV, and the integrals are approximated by algebraic expressions involving the values of the unknown variables at specific discrete locations (e.g. the CV centres). This results in a system of algebraic equations whose solution gives approximate values for the unknowns at these locations. In general, the finer the grid the smaller the discrepancy between the algebraic and the differential equations, and the better the results. On the other hand, using a higher grid density also increases the computational effort required. Therefore, for given computational resources, it is desirable to have a way of determining the optimal grid density distribution in space which results in the maximum possible accuracy.

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Some methods move the existing nodes of the grid to new locations in order to improve the accuracy of the results without altering the number of nodes of the grid (an interesting example is [1], based on truncation error). However, the most popular methods are local refinement methods which are more easily applicable to more general grids and complex geometries and which locally increase the grid density by subdividing selected CVs into smaller CVs. Various grid refinement criteria have been proposed which try to identify the CVs whose subdivision would be most beneficial in terms of increase of accuracy, but there is no global agreement among researchers as to which criterion is optimal.

In the context of the finite volume method, refining CVs where the gradient of a selected flow variable is large appears to have been popular in the past (e.g. [2,3]). This simple choice is useful for capturing flow features such as fronts or shock waves, but otherwise it is not appropriate; second- or higher-order discretisation schemes should be able to capture these gradients no matter how high without special grid refinement, as long as the higher-order derivatives of the flow variables are small. Also, such a method could potentially not converge to the exact solution of the partial differential equation as convergence may require also the refinement of CVs where the gradients are small but the higher-order derivatives are large.

Another popular class includes local refinement criteria which are based on the truncation error (e.g. [4–12]). This has a more solid theoretical background since the truncation error acts as the source of the discretisation error (see Section 2). An interesting alternative to the use of the truncation error is the use of the *residual* (e.g. [13,8,14–19]): While the truncation error is obtained by applying the discrete finite volume operator to the exact solution, the residual is obtained by applying the exact differential operator to the approximate finite volume solution. These two quantities have similar properties. Yet another method is to base the local refinement criterion on an estimate or indicator of the discretisation error itself (e.g. [20,21]), but as discussed in [16,18] this is not efficient since the discretisation error is convected and diffused in the domain and may be high in regions which do not really need a higher grid density. Grid refinement indirectly reduces the discretisation error by reducing the *source* of this error (which can be viewed to be either the truncation error or the residual) – and so it is more appropriate to base the refinement criterion on the source.

The present study attempts a detailed evaluation of the efficiency of truncation error-based local refinement for the Navier–Stokes equations, by testing and comparing various truncation error-based criteria on a lid-driven cavity problem. Although the aforementioned studies demonstrate that truncation error-guided local refinement can be beneficial, such a detailed study is missing, to the authors' knowledge. A possible explanation for this is that it is difficult to quantify the efficiency (the ratio of the discretisation error reduction to the number of control volumes added to the grid), as the Navier–Stokes equations do not have analytic solutions except for very simple cases, and so it is difficult to calculate the discretisation error accurately. In the present work a variant of the lid-driven cavity test case is selected (by far the most popular test case for incompressible flows), and the problem is first solved on very fine structured grids for various Reynolds numbers. Using Richardson extrapolation "exact" solutions are obtained, and the various local refinement techniques are evaluated by comparing the discretisation error against the number of CVs of the locally refined grids in each case. Sections 2 and 3 present some theory concerning the truncation error, its use as a refinement criterion, and a method to estimate it *a posteriori*. Section 4 describes the specific finite volume method used to solve the incompressible Navier–Stokes equations. Finally, Sections 6 and 7 present the lid-driven cavity test case and the results.

2. The truncation error as a grid refinement criterion

The truncation error quantifies the discrepancy between the differential equation integrated over a CV and the approximate algebraic equation which is obtained for that CV via the finite volume method. This makes it appear to be the right quantity to use as a refinement criterion, and indeed many researchers have used it as such (e.g. [4–12]), but there are some complexities. At this point it is appropriate to include a brief description of the truncation error and of some of its properties that are of interest.

Suppose a finite volume method is used to solve the (possibly non-linear) partial differential equation (PDE) $N(u) = b$ where N is the differential operator, u is the unknown function, and b is a known function independent of u. This equation is integrated over each CV and the result is approximated by an algebraic equation. For the whole grid one obtains a system of algebraic equations:

$$\left[\frac{1}{\delta\Omega_P}\int_{\delta\Omega_P} N(u)\,d\Omega\right]_{P=1}^{K} = N_h(u_h) + \tau_h \tag{1}$$

Both sides of (1) are vectors with K components, where K is the number of CVs in the grid. On the left-hand side, each component P is the integral of the image $N(u)$ over CV P divided by the CV volume $\delta\Omega_P$. On the right-hand side the term $N_h(u_h)$ is a vector which is obtained by applying the algebraic operator N_h to the vector u_h of the values of the exact solution function u at the CV centres. The symbol h is sometimes used to denote a particular grid, and sometimes used to denote the spacing between successive grid lines of that grid – this will be clear from the context. N_h is constructed by discretising the continuous differential operator N according to the finite volume methodology using selected discretisation schemes. No matter how accurate these discretisation schemes are, it is in general not possible for the term $N_h(u_h)$ to equal the left hand side, but they will differ by the truncation error τ_h. The truncation error consists of all the terms that were truncated from the Taylor series used in converting the differential equations to algebraic form (see e.g. [22]).