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Numerical computation of the helical Chandrasekhar–Kendall modes

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ABSTRACT

A new formulation is presented for numerically computing the helical Chandrasekhar–Kendall modes in an axisymmetric torus. It explicitly imposes $\nabla \cdot \mathbf{B} = 0$ and yields a standard matrix eigenvalue problem, which can then be solved by standard matrix eigenvalue techniques. Numerical implementation and computational results are shown for an axisymmetric torus typical of reversed field pinch and spherical tokamak.

1. Introduction

The eigenfunctions of the curl operator are called Beltrami functions or Beltrami-Trkalian functions. They have found wide-spread use in fluid dynamics, plasma physics, and engineering. Yoshida [1] gave a list of specific applications with references. These eigenfunctions in the name of Chandrasekhar–Kendall (CK) modes [2] for force-free magnetic field received special attention in magnetized plasma physics due to the seminal work by Woltjer [3] and Taylor [4] that force-free magnetic field is a natural state of a relaxed plasma which reaches a minimum magnetic energy state while conserving the magnetic helicity. In addition to relaxation theory, they have also been profitably used as spectral basis for hydrodynamic and magneto-hydrodynamic computation of a dynamical plasma [5,6], largely due to their completeness as functional basis.

Chandrasekhar–Kendall modes [2] are eigensolutions to the force-free equation of the magnetic field $\mathbf{B}$,

$$\nabla \times \mathbf{B} = k\mathbf{B}, \quad \mathbf{B} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0, \tag{1}$$

with homogeneous boundary conditions, which are equivalent to a vanishing vacuum magnetic field inside the discharge chamber Ω. These force-free eigensolutions are uniquely determined by the chamber geometry, and play an essential role in determining the relaxed states of a driven plasma [4,7], in which the spatial overlap of the CK modes and a vacuum field generated by external current provides the coupling between the external helicity source and the driven plasma [8]. In fact, resonant coupling [9–12] is the physical mechanism underlying the self-organization of system-scale magnetic fields by magnetic relaxation in the laboratory formation of spherical tokamak [13,14], spheromak [15], and reversed field pinch by helicity injection. It is also thought to be a competing paradigm for the generation and sustainment of large scale magnetic fields in astrophysical radio lobes [16].

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Although much of the design constraint and optimization for laboratory helicity injection applications involves only the axisymmetric CK modes in an axisymmetric toroidal chamber [17,14,15], as the target magnetic configurations are axisymmetric [18], helical or non-axisymmetric CK modes can play a subtle role in determining the operating boundary and the degrees of intrinsic non-axisymmetry in the experiments. This comes about in two ways. The first is the resonant coupling between non-axisymmetric error fields and the helical CK modes, which leads to an amplified helical field in a relaxed driven plasma. The second is the possibility of a Taylor's mixed state [7], in which a single helical CK mode provides the sink for arbitrarily large amount of externally supplied magnetic helicity. Although a pure Taylor's mixed state with constant k is unlikely to be obtained experimentally, the operating boundary suggested by it does imply the onset of robust helical instabilities that could saturate into large helical magnetic fields.

In both cases, the helical CK modes and their eigenvalues are required to quantify the magnetic configuration of the relaxed states. It turns out that numerical solution of helical CK modes requires considerably more effort than their axisymmetric counterparts, which can be solved from the force-free Grad–Shafranov equation as a scalar eigenvalue problem with homogeneous Dirichlet boundary conditions. The objective of this paper is to show a new formulation of the helical CK eigenmode equations in an axisymmetric toroidal chamber and present the numerical solutions in an axisymmetric torus typical of spherical tokamak and reversed field pinch configurations. The primary result of the paper is a new formulation that numerically computes the helical CK modes as a standard matrix eigenvalue problem, which is suitable for numerical implementation in a torus of complex poloidal cross section. This capability can be of use for current and future design optimizations for reversed field pinch, spherical tokamak [14], and spheromak [15] experiments, and for astrophysical radio lobe analysis [16].

2. Previous calculation of helical CK modes

2.1. Chandrasekhar–Kendall scalar function formulation

A number of earlier calculations of axisymmetric and helical CK modes were carried out using a scalar function formulation of a Taylor state magnetic field. Chandrasekhar and Kendall [2] noted that a force-free field in a Taylor state can be written as

$$\mathbf{B} = \nabla \times (\hat{\mathbf{a}}\psi) + \frac{1}{k}\nabla \times \nabla \times (\hat{\mathbf{a}}\psi) = -\hat{\mathbf{a}} \times \nabla\psi - \frac{1}{k}\nabla \times (\hat{\mathbf{a}} \times \nabla\psi), \tag{2}$$

with $\hat{\mathbf{a}}$ a fixed (does not vary in space) unit vector. For example, $\hat{\mathbf{a}}$ can be $\hat{\mathbf{Z}}$ in a cylindrical coordinate system. Unit vector $\hat{\mathbf{R}}$, in contrast, does not satisfy the requirement. The constraint for a Taylor state solution is for ψ to satisfy a scalar Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0. \tag{3}$$

There is a long history on understanding the solution space of the eigenfunctions of the curl operator [19] and the completeness of the CK modes in a cylindrical geometry [1] and a spherical geometry [20]. Numerical evaluation of these modes in a complex geometry where the analytical expression is not available has some unusual challenges. This can be seen by examining the boundary condition for $\mathbf{B}$.

Let $\hat{\mathbf{n}}$ be the unit vector normal to the boundary, the boundary condition $\mathbf{B} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0$ implies

$$\left[\hat{\mathbf{n}} \times \hat{\mathbf{a}} \cdot \nabla\psi + \frac{1}{k}\hat{\mathbf{n}} \cdot \nabla \times (\hat{\mathbf{a}} \times \nabla\psi)\right]|_{\partial\Omega} = 0. \tag{4}$$

One can compute the helical Taylor states in an axisymmetric chamber by using Eqs. (2) and (3). The complication in imposing the boundary condition can be illustrated in a torus with rectangular poloidal cross section. One can let $\mathbf{a} = \hat{\mathbf{Z}}$, so

$$\hat{\mathbf{n}} \times \hat{\mathbf{Z}} \cdot \nabla\psi = \frac{1}{R}\frac{\partial\psi}{\partial\varphi}, \quad \text{for } \hat{\mathbf{n}} = \hat{\mathbf{R}},$$

and

$$\hat{\mathbf{n}} \times \hat{\mathbf{Z}} \cdot \nabla\psi = 0, \quad \text{for } \hat{\mathbf{n}} = \hat{\mathbf{Z}}.$$

Recall that in cylindrical coordinates,

$$\nabla \times (\hat{\mathbf{Z}} \times \nabla\psi) = -\frac{\partial^2\psi}{\partial R\partial Z}\hat{\mathbf{R}} - \frac{1}{R}\frac{\partial^2\psi}{\partial\varphi\partial Z}\hat{\boldsymbol{\varphi}} + \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) - \frac{1}{R^2}\frac{\partial^2\psi}{\partial\varphi^2}\right]\hat{\mathbf{Z}}.$$

In a cylinder, the $\hat{\mathbf{Z}} \cdot \mathbf{B}$ vanishes at the top and bottom ($Z = 0, L$), which implies

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) - \frac{1}{R^2}\frac{\partial^2\psi}{\partial\varphi^2} = 0.$$