Journal of Computational Physics 230 (2011) 1147–1170

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier–Stokes equations

N.C. Nguyen [a,*], J. Peraire [a], B. Cockburn [b]

[a] Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[b] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

ARTICLE INFO

Article history:
Received 2 January 2010
Received in revised form 20 October 2010
Accepted 25 October 2010
Available online 2 November 2010

Keywords:
Finite element methods
Discontinuous Galerkin methods
Hybrid/mixed methods
Augmented Lagrangian
Incompressible Navier–Stokes equations

ABSTRACT

We present an implicit high-order hybridizable discontinuous Galerkin method for the steady-state and time-dependent incompressible Navier–Stokes equations. The method is devised by using the discontinuous Galerkin discretization for a velocity gradient-pressure–velocity formulation of the incompressible Navier–Stokes equations with a special choice of the numerical traces. The method possesses several unique features which distinguish itself from other discontinuous Galerkin methods. First, it reduces the globally coupled unknowns to the approximate trace of the velocity and the mean of the pressure on element boundaries, thereby leading to a significant reduction in the degrees of freedom. Moreover, if the augmented Lagrangian method is used to solve the linearized system, the globally coupled unknowns become the approximate trace of the velocity only. Second, it provides, for smooth viscous-dominated problems, approximations of the velocity, pressure, and velocity gradient which converge with the optimal order of $k+1$ in the L^2-norm, when polynomials of degree $k \geqslant 0$ are used for all components of the approximate solution. And third, it displays superconvergence properties that allow us to use the above-mentioned optimal convergence properties to define an element-by-element postprocessing scheme to compute a new and better approximate velocity. Indeed, this new approximation is exactly divergence-free, $\boldsymbol{H}(\mathrm{div})$-conforming, and converges with order $k+2$ for $k \geqslant 1$ and with order 1 for $k=0$ in the L^2-norm. Moreover, a novel and systematic way is proposed for imposing boundary conditions for the stress, viscous stress, vorticity and pressure which are not naturally associated with the weak formulation of the method. This can be done on different parts of the boundary and does not result in the degradation of the optimal order of convergence properties of the method. Extensive numerical results are presented to demonstrate the convergence and accuracy properties of the method for a wide range of Reynolds numbers and for various polynomial degrees.

1. Introduction

In recent years, discontinuous Galerkin (DG) methods have emerged as a competitive alternative for numerically solving the incompressible Navier–Stokes equations [3,18,19,32,44]. The advantages of the DG methods over classical continuous Galerkin finite element, finite difference and finite volume methods are well-documented in the literature (see [5,6,21] and references therein): the DG methods work well on arbitrary meshes, result in stable high order discretizations of the convective and diffusive operators, allow for a simple and unambiguous imposition of boundary conditions and are well suited for parallelization and adaptivity. Despite all these advantages, there are still obstacles which prevent DG methods

* Corresponding author. Tel.: +1 617 253 8080.
E-mail address: cuongng@mit.edu (N.C. Nguyen).

0021-9991/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2010.10.032

from becoming the method of choice for a wide class of applications. One such obstacle is the high computational cost associated with DG methods which can be traced to the larger number of globally coupled unknowns when compared to continuous Galerkin finite elements, finite differences, or finite volume schemes.

In this paper, we introduce a hybridizable discontinuous Galerkin (HDG) method for the numerical solution of the incompressible Navier–Stokes equations. The HDG methods were first introduced for diffusion [13] and continuum mechanics [26,45] problems. They were analyzed in [7,15,17], see also [16], in the setting of diffusion problems, and then developed for linear and nonlinear convection–diffusion problems [8,37,38,46], Stokes flow [4,11,14,39,40]. The HDG method for the compressible Euler and Navier–Stokes equations is introduced in [43]. An overview of recent development of HDG methods is provided in [41]. The HDG methods retain the advantages of standard DG methods and provide a significantly reduced number of globally-coupled degrees of freedom, thereby allowing for a substantial reduction in the computational cost.

In this paper, we devise the first HDG method for the incompressible Navier–Stokes equations by extending the HDG methods for convection–diffusion [37,38] and the HDG methods for the Stokes system [14,39,40], and showing that the distinctive advantages of those HDG methods are retained, namely:

- **Reduced number of degrees of freedom**. Unlike *all* known other DG methods, which result in a final system involving the degrees of freedom of the approximate velocity and pressure, the HDG method produces a final system involving the degrees of freedom of the *approximate trace* of the velocity and the *mean* of pressure. Since the approximate trace is defined on the element borders only and since the mean of pressure is a piece-constant function, the HDG method has significantly less globally coupled unknowns than other DG methods, especially for high-degree polynomial approximations. Moreover, if the augmented Lagrangian method [23] is used to solve the linearized system, the globally coupled unknowns become the approximate trace of the velocity only. This large reduction in the degrees of freedom leads to significant savings for both computational time and memory storage.
- **Optimal convergence**. The HDG method provides an approximate velocity, pressure and velocity gradient converging with the optimal order $k + 1$ in the L^2-norm for viscous-dominated flows with smooth solution; here k is the degree of the polynomials used to represent all components of the approximate solution. This has to be contrasted with the fact that *all* known DG methods display the suboptimal order of convergence of k for the approximate pressure and for the velocity gradient or the vorticity. This includes, the first DG method for the Navier–Stokes equations [28,29], as well as the family of DG methods for the Navier–Stokes equations proposed in [18,19].
- **Superconvergence and local postprocessing**. The HDG method has superconvergence properties for the velocity which, combined with the above-mentioned optimal converge properties, allows us to use an element-by-element postprocessing, proposed in [14] for HDG methods for Stokes flow, to obtain a new and better approximation of the velocity. Unlike the original velocity, the postprocessed velocity is exactly divergence-free, $\boldsymbol{H}(\text{div})$-conforming, and converges with order $k + 2$ for $k \geqslant 1$. Since the postprocessing is performed at the element level, the computational cost involved in obtaining the postprocessed velocity is very small.
- **Unified treatment of boundary conditions and the numerical fluxes.** The HDG method entails a single numerical flux formulae containing both the viscous and inviscid numerical fluxes. Different boundary conditions can be included in a single framework by defining appropriate numerical fluxes on the boundaries of the physical domain. This *novel and systematic* manner of imposing boundary conditions allows for pressure, vorticity and stress boundary conditions to be prescribed on different parts of the boundary.

Let us briefly emphasize the fact that, to the knowledge of the authors, no other known DG or mixed method for the Navier–Stokes equations has all the above four properties of these HDG methods. Note that, as we pointed out in [40], some DG methods provide velocities that are divergence-free inside each of the element which, however, do not lie on $\boldsymbol{H}(\text{div})$ since their normal component has no interelement continuity. Examples are the first DG method proposed for the Stokes system [2] and for the Navier–Stokes equations in [28] and, more recently, and the DG method for the Stokes equations proposed in [33] and in [34] for the Navier–Stokes equations. Note also that there are DG methods that do provide velocities that are divergence-free and belong to $\boldsymbol{H}(\text{div})$. A wide family DG methods with this property were introduced in [18] for the Navier–Stokes equations, even though only a particular case was treated in detail therein. Other particular cases were developed later in [19] and (for the Stokes problem) in [47,49]; the latter method was then extended to the Navier–Stokes equation in [48]. Finally, note that the first DG method whose formulation involves $\boldsymbol{H}(\text{div})$-conforming velocities with are also divergence-free was proposed in [4] for the Stokes equations; see also [9,10] for an extension of this approach to a mixed method. However, their velocities converge with order at most $k + 1$ for $k \geqslant 1$.

Recently, there have been new developments in DG methods—the multiscale discontinuous Galerkin (MDG) method [27] and the embedded discontinuous Galerkin method (EDG) [13,26]—which aim to reduce the globally coupled degrees in a DG discretization. Like the HDG method, these DG methods solve for the approximate trace of the field variables. However, unlike the HDG method, the approximate trace in these methods resides in a C^0 space. Therefore, neither the MDG nor the EDG method have the local conservativity property of the HDG method. As a consequence, see [16], these DG methods do not have some superconvergence properties of the HDG method and hence their approximate solution can not be postprocessed to yield a higher-order convergent approximation. Further development of the MDG method for the incompressible Navier–Stokes equations leads to the so-called Galerkin interface stabilisation (GIS) method [31]. As expected, the GIS method provides suboptimal convergence for the approximate pressure and stress even for the Stokes system.