



Approximation of the thermally coupled MHD problem using a stabilized finite element method

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ABSTRACT

A numerical formulation to solve the MHD problem with thermal coupling is presented in full detail. The distinctive feature of the method is the design of the stabilization terms, which serve several purposes. First, convective dominated flows in the Navier–Stokes and the heat equation can be dealt with. Second, there is no restriction in the choice of the interpolation spaces of all the variables and, finally, flows highly coupled with the magnetic field can be accounted for. Different aspects related to the design of the final fully discrete and linearized algorithm are also discussed.

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1. Introduction

The objective of this work is to present a finite element method for the approximation of the thermally-coupled magneto-hydrodynamic (MHD) problem. We discuss several issues related to the time discretization, the linearization and the iterative coupling of the unknowns. However, our main concern is the design of a stabilization technique that allows one to use any continuous interpolation for all the unknowns, in particular, equal interpolation is allowed.

Considering first the thermally uncoupled case, in principle the unknowns involved are the magnetic field, the fluid velocity and the hydrodynamic pressure. However, to enforce the divergence free condition for the numerical approximation of the magnetic field we introduce a magnetic pseudo-pressure (whose exact value should be zero). This zero divergence condition is automatically satisfied at the continuous level for the transient problem if the initial magnetic field is solenoidal, but it is convenient to explicitly enforce it in the numerical approximation, especially for stationary problems. With the introduction of the magnetic pseudo-pressure we are left with a system of four equations with four unknowns. The ‘augmented’ approach used in this work is discussed for example in [19] in the context of MHD and more recently in [5] for the Maxwell equations. The same approach is used in [14,29,24,23,28] (see also references therein). Other possibilities of enforcing the divergence free condition for the magnetic field are penalty strategies (see for example [1]) or the use of (weakly) divergence free interpolations based on Nédélec-type elements (described for example in [25,26]). These elements can also be used in combination with the augmented approach using a continuous approximation for the magnetic pseudo-pressure [28] so as to satisfy the adequate inf–sup condition between this scalar and the magnetic field. This condition also holds if an equal order discontinuous interpolation is used for both variables [17]. Nevertheless, there is also the possibility of relying on the mathematical structure of the equations and to expect that the original problem will already yield a magnetic field close

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enough to solenoidal. This is the idea followed in [22], which probably contains the first analysis of a finite element approximation to the MHD problem, and it is also used in [15], among other papers.

Having introduced the magnetic pseudo-pressure as a new unknown in the problem, its finite element approximation has several difficulties. First, there is the well known compatibility condition between the approximation spaces for the velocity and the pressure, but also for the approximation spaces for the magnetic field and the magnetic pseudo-pressure. Both conditions can be expressed in a standard inf–sup form [2]. There is also the problem of dealing with situations in which first order derivatives, both in the Navier–Stokes equations and in the equation for the magnetic field, dominate (from the numerical point of view) the second order terms that give an elliptic nature to the system of equations to be solved. These are the classical convection-dominated flow problems. Both the compatibility condition between interpolating spaces and the oscillations found in flows dominated by convection can be overcome by using stabilized finite element methods. First approaches in this direction can be found in [15] (without the introduction of the magnetic pseudo-pressure) and in [24,7] (where the magnetic pseudo-pressure is also introduced). However, another particular feature of the MHD problem are the couplings involved. In the magnetic problem, the coupling with the hydrodynamic problem comes from the convective term in the equation for the magnetic field, whereas in the Navier–Stokes equations the coupling with the magnetic problem comes from Lorentz’s force. Our objective is to design a stabilized finite element method that takes these couplings into account.

The stabilized finite element method presented here is based on the two-scale decomposition of the unknowns into their finite element component and a subscale that cannot be captured by the finite element space. The format that we follow of this idea was introduced in [18]. In particular, the version for systems we employ here was already presented in [6]. A first version of our formulation, considering only the stationary and thermally uncoupled problem, can be found in [9].

The formulation is first designed for linear problems, and therefore our first concern is to devise a linearization technique for the fully coupled problem. For simplicity, we consider a fixed point method. Among the different possibilities, we identify the only one that leads to a linearized problem that is coercive, and thus guarantees existence and uniqueness of solution. This fixed point method is often used, but rarely justified. It is for this linearized problem that we propose a stabilized finite element method based on the subgrid scale concept. The important point is how to approximate the subgrid scales. We use the simplest approach of taking them proportional to a projection of the residual of the finite element approximation multiplied by the so called matrix of stabilization parameters. We consider two possibilities for the projection. The first is to take it as the identity (at least when applied to the residual of the finite element solution), and the second is to consider this projection as the orthogonal to the finite element space. The first option leads to a classical residual based stabilized finite element method, whereas the second was termed orthogonal subscale stabilization formulation (OSS) in [7], where it is fully developed for incompressible flows. A thorough numerical analysis for the stationary and linearized problem can be found in [8].

The design of the matrix of stabilization parameters is solely based on the stability and convergence analysis of the problem. This analysis will be presented in a situation as simple as possible, trying to avoid mathematical technicalities. It is not the purpose of this paper a deep numerical analysis of the formulation to be presented, but to present it with a sound motivation. The resulting formulation differs from the one proposed in [24] both in the structure of the stabilizing terms (no attempt is made there to account neither for convection-dominated situations nor for the coupling effects) and in the design of the stabilization parameters. It also differs from the method proposed in [15] in the inclusion of the magnetic pseudo-pressure and in the design of the stabilization parameters.

The paper is organized as follows. The problem to be solved is presented in the Section 2, including its strong and its variational forms. Issues not directly related to the finite element approximation are treated in Section 3, where a simple time integration scheme is described and linearization possibilities are discussed, starting with the identification of the only feasible fixed-point iteration for the thermally uncoupled MHD problem and then including the thermal coupling. The stabilization method is proposed and fully analyzed for the linearized stationary MHD problem in Section 4. The scheme we finally propose is written in Section 5. Numerical examples are presented in Section 6 and conclusions are finally drawn in Section 7.

2. Problem statement

2.1. Initial and boundary value problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a domain where we want to solve the thermally coupled MHD problem during the time interval $[0, T]$, the thermal coupling being modeled through Boussinesq’s assumption. The unknowns of the problem are the fluid velocity $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$, the pressure $p : \Omega \times (0, T) \rightarrow \mathbb{R}$, the magnetic induction (which we will simply call magnetic field) $\mathbf{B} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$, the magnetic pseudo-pressure $r : \Omega \times (0, T) \rightarrow \mathbb{R}$ and the temperature $\vartheta : \Omega \times (0, T) \rightarrow \mathbb{R}$, which are solution of the system of partial differential equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p - \frac{1}{\mu_m \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{g} \beta \vartheta = \mathbf{f}_f + \mathbf{g} [1 + \beta \vartheta_r], \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

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