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A splitting approach for the fully nonlinear and weakly dispersive Green–Naghdi model

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ABSTRACT

The fully nonlinear and weakly dispersive Green–Naghdi model for shallow water waves of large amplitude is studied. The original model is first recast under a new formulation more suitable for numerical resolution. An hybrid finite volume and finite difference splitting approach is then proposed, which could be adapted to many physical models that are dispersive corrections of hyperbolic systems. The hyperbolic part of the equations is handled with a high-order finite volume scheme allowing for breaking waves and dry areas. The dispersive part is treated with a classical finite difference approach. Extensive numerical validations are then performed in one horizontal dimension, relying both on analytical solutions and experimental data. The results show that our approach gives a good account of all the processes of wave transformation in coastal areas: shoaling, wave breaking and run-up.

1. Introduction

In an incompressible, homogeneous, inviscid fluid, the propagation of surface waves is governed by the Euler equations with nonlinear boundary conditions at the surface and at the bottom. In its full generality, this problem is very complicated to solve, both mathematically and numerically. This is the reason why more simple models have been derived to describe the behavior of the solution in some physical specific regimes. A recent review of the different models that can be derived can be found in [23].

Of particular interest in coastal oceanography is the *shallow-water* regime, which corresponds to the configuration where the wave length λ of the flow is large compared to the typical depth h_0:

$$\text{(Shallow water regime)}\ \mu := \frac{h_0^2}{\lambda^2} \ll 1.$$

When the typical amplitude a of the wave is small, in the sense that

$$\text{(Small amplitude regime)}\ \varepsilon := \frac{a}{h_0} = O(\mu)$$

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it is known that an approximation of order $O(\mu^2)$ of the free surface Euler equations is furnished by the Boussinesq systems, such as the one derived by Peregrine in [30] for uneven bottoms. This model couples the surface elevation ζ to the vertically averaged horizontal component of the velocity V, and can be written in non-dimensionalized form as

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hV) = 0, \\ \partial_t V + \varepsilon(V\cdot\nabla)V + \nabla\zeta = \mu\mathcal{D} + O(\mu^2), \end{cases} \tag{1}$$

where h is the water depth and $\mathcal{D}$ accounts for the nonhydrostatic and dispersive effects, and is a function of ζ, V and their derivatives. For instance, in the Boussinesq model derived in [30], one has

$$\mathcal{D} = \frac{h}{2}\nabla[\nabla\cdot(h\partial_t V)] - \frac{h^2}{6}\nabla^2\partial_t V. \tag{2}$$

Unfortunately, the small amplitude assumption $\varepsilon = O(\mu)$ is too restrictive for many applications in coastal oceanography, where *large amplitude* waves have to be considered,

$$(\text{Large amplitude regime})\ \varepsilon := \frac{a}{h_0} = O(1).$$

If one wants to keep the same $O(\mu^2)$ precision of (1) in a large amplitude regime, then the expression for $\mathcal{D}$ is much more complicated than in (2). For instance, in 1D and for flat bottoms, one has

$$\mathcal{D} = \frac{\mu}{3h}\partial_x\left[h^3(V_{xt} + \varepsilon V V_{xx} - \varepsilon(V_x)^2)\right].$$

In this regime, the corresponding Eq. (1) have been derived first by Serre and then Su and Gardner [34], Seabra-Santos et al. [32] and Green and Naghdi [18] (other relevant references are [5,14,29,40]); consequently, these equations carry several names: Serre, Green–Naghdi, or fully nonlinear Boussinesq equations. We will call them *Green–Naghdi equations* throughout this paper. Here again, we refer to [23] for more details; note also that a rigorous mathematical justification of these models has been given in [1].

The Green–Naghdi equations (1) provide a correct description of the waves up to the breaking point; from this point however, they become useless (at least without consequent modifications). A first approach to model wave breaking is to add an ad hoc viscous term to the momentum equation, whose role is to account for the energy dissipation that occurs during wave breaking. This approach has been used for instance by Zelt [43] or Kennedy et al. [21] and Chen et al. [13]. Recently, Cienfuegos et al. [16] proposed a new 1D wave-breaking parametrization including viscous-like effects on both the mass and the momentum equations. This approach is able to reproduce wave height decay and intraphase nonlinear properties within the entire surf zone. However, the extension of this ad hoc parametrization to 2D wave cases remains a very difficult task. Another approach to handle wave breaking is to use the classical nonlinear shallow water equations, defined with $\mathcal{D} = 0$ in (1) and denoted by NSWE in the following. These equations being hyperbolic, they develop shocks; after the breaking point, the waves are then described by the *weak solutions* of this hyperbolic system. This approach, used in [4,22] is satisfactory in the sense that it gives a natural and correct description of the dissipation of energy during wave breaking. Its drawback, however, is that it is inappropriate in the shoaling zone since this models neglects the nonhydrostatic and dispersive effects. The motivation of this paper is to develop a model and a numerical scheme that describes correctly both phenomena. More precisely, we want to

♯1 Provide a good description of the dispersive effects (in the shoaling zone in particular).

♯2 Take into account wave breaking in a simple way.

Another theoretical and numerical difficulty in coastal oceanography is the description of the shoreline, i.e. the zone where the water depth vanishes, as the size of the computational domain becomes part of the solution. Taking into account the possibility of a vanishing depth while keeping the dispersive effects is more difficult, see [15,41] for instance. As for the breaking of waves, neglecting the nonhydrostatic and dispersive effects makes the things simpler. Indeed, various efficient schemes have been developed to handle the possibility of vanishing depth for the NSWE with source terms, relying for instance on coordinates transformations [7], artificial porosity [38] or even variables extrapolations [24]. In a simpler way, it is shown in [26] that the occurrence of dry areas can be naturally handled with a water height positivity preserving finite volume scheme, without introducing any numerical trick. Of course, the price to pay is the same as above: the dispersive effects are lost. Hence the third motivation for this paper:

♯3 Propose a simple numerical method that allows at the same time the possibility of vanishing depth and dispersive effects.

The strategy adopted here to handle correctly the three difficulties ♯ 1–3 identified above starts from the Green–Naghdi equations. As already said, they are very well adapted to ♯ 1. With a careful choice of the numerical methods, they also allow for the possibility of vanishing depth, and thus answer to ♯ 3. The main difficulty is thus to handle ♯ 2 (i.e. wave breaking) with a code based on the Green–Naghdi equation. In order to do so, we use a numerical scheme that decomposes the hyperbolic and dispersive parts of the equations [17]. We also refer to [25] for a recent numerical analysis of the Green–Naghdi equations based on a Godunov type scheme and that provide good results for the dam break problem. We use here a second