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A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions ☆

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ABSTRACT

One-dimensional fractional anomalous sub-diffusion equations on an unbounded domain are considered in our work. Beginning with the derivation of the exact artificial boundary conditions, the original problem on an unbounded domain is converted into mainly solving an initial-boundary value problem on a finite computational domain. The main contribution of our work, as compared with the previous work, lies in the reduction of fractional differential equations on an unbounded domain by using artificial boundary conditions and construction of the corresponding finite difference scheme with the help of method of order reduction. The difficulty is the treatment of Neumann condition on the artificial boundary, which involves the time-fractional derivative operator. The stability and convergence of the scheme are proven using the discrete energy method. Two numerical examples clarify the effectiveness and accuracy of the proposed method.

1. Introduction

As a subset of the class of pseudo-differential equations, fractional partial differential equations (FPDEs) have become a hot topic in the fields of both theoretical studies and applicable engineering. The physical backgrounds cover describing dynamical properties of linear viscoelastic media, modeling the relaxation in polymer systems, electrical circuit called fractance, dynamics in control system, groundwater hydrology, the anomalous diffusion behaviors of particles and so on (see [1–6]), most of which are with properties of long memory and non-local dependence. For the Brownian motion of particles, the mean square displacement varies linearly with respect to time (Fick's law) and the corresponding equation is the standard diffusion equation. Anomalous diffusion phenomenon has been observed early in 1926 by Richardson in treating the turbulence diffusion problem. Since then, amounts of physical phenomenons and experiments verify the existence of anomalous diffusion in various fields, including porous systems, nuclear magnetic resonance, transport on fractal geometries and some others [7]. An efficient way displaying this behavior is the fractional calculus and equations related on it are fractional diffusion equations, which describe the probability density of particles diffusing with mean square displacement $\chi^2(t) \propto t^{\gamma}$ and sub-diffusion corresponds to $0 < \gamma < 1$, while the super-diffusion with $1 < \gamma < 2$.

In the last few decades, more and more scholars have realized FPDEs' significant values in depicting the real phenomenons and the field has gained a huge development. A large amount of fractional models have been proposed and investigated in various aspects such as the advantages over the integer-order counterparts, analytical results, numerical techniques and so on.

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The problems in finite domains have been widely studied from different aspects. For the existence and analytical solutions to FPDEs in finite domains, abundant literatures have discussed these issues. We refer to readers [1–6] and references therein. From the numerical viewpoint, the FPDEs have been extensively concerned. Many powerful methods have been proposed, such as finite difference method (FDM) [8–27], finite element method (FEM) [28,29], random walk approach (RWA) [30], spectral method (SM) [31,32], the decomposition method (DM) [33,34], the homotopy perturbation method (HPM) [35,36], the integral equation method (IEM) [37], reproducing kernel method (RKM) [38], the variational iteration method (VIM) [39] and so many others.

Many scholars are today concerned with finite difference methods for solving FPDEs. For the Riemann–Liouville fractional derivative operator, most of the approximations are based on the Grünwald–Letnikov discretization, where only first-order accuracy is achieved usually, see the monographs [9–11,13,14,16,17,25] and the relative works. Wang et al. [10] developed a fast characteristic difference method for solving space-fractional advection diffusion equation, which combined the characteristic difference method for the integer-order analog and shifted Grünwald–Letnikov discretization for the fractional approximation together. The fast property of this algorithm lies in the reduced amounts of storage and computation by considering the features of coefficient matrix. The other way to approximate the fractional operators relies on the interpolation discretization, including L1 scheme [1,12,17–19,22–24,26,27], L2 scheme [1,12,15], L2C scheme [1,15] and so on. For the time-fractional anomalous diffusion equations, Murillo and Yuste [13] compared three explicit difference methods, where all three methods were based on the Grünwald–Letnikov discretization. Zhuang et al. [18] constructed a finite difference scheme based on L1 discretization and analyzed the stability and convergence by maximum principle. Liu et al. [22] developed a meshless approach method using L1 approximation for time fractional derivative and radial basis function (RBF) approximation for spatial discretization. A novel technique, which first integrated the original equation on both hand sides and then approximated the obtained equivalent form using the ideas of numerical integrals, has been applied in [20]. Sun and Wu [23] constructed a finite difference scheme for fractional diffusion-wave system. Langlands and Henry [24] discussed a fractional diffusion equation with Neumann boundary conditions. Cui [25], Du et al. [26] and Gao et al. [27] presented the spatial high accuracy schemes for fractional sub-diffusion and super-diffusion equations. Lin and Xu [40] combined the L1 approximations for time fractional part and spectral approximations for spatial part together. In [41], the authors considered the time fractional cable equations by using the same L1 discretization for time fractional derivatives.

The research of fractional-order problems on unbounded domains is also of great importance. There were several literatures discussing the relevant analytical solutions. The literatures [42–44] investigated the existence of solutions of the boundary value problem on unbounded domains, based on the fixed point theorem combined with the diagonalization method, for a class of fractional ordinary differential equations (FODEs) involving the Caputo fractional derivative and the Riemann–Liouville fractional derivative. Huang [45] discussed the time fractional telegraph equation on both a whole-space domain and a half-space domain by means of Laplace and Fourier transforms. Langlands et al. [46] derived the solutions of fractional cable equations with different boundary conditions on infinite domains by the similar techniques, where the solution was expressed using Fox *H* functions that seemed complex. Valkó and Zhang [47] listed the fundamental solution of the one-dimensional space–time fractional diffusion equation. Mainardi [48], Gorenflo et al. [49] derived the fundamental solutions for the Cauchy problem and signalling problem of fractional diffusion equation using the Laplace transform and the properties of *M*-Wright function. There are few works on the numerical methods of FPDEs on unbounded domains in spite of their importance in practical applications. The works on this respect are still in the initial stage and considerable aspects of this issue need to be explored.

The great difficulties to obtain the numerical solutions of these problems on unbounded physical domains lie in the unboundedness of physical domains [50–66]. Finite difference methods (FDMs) and finite element methods (FEMs) cannot be used directly. Numbers of scholars have agreed that the artificial boundary methods (ABMs) are one of the most popular ways to deal with the problems on unbounded physical domains, by which, the problem on an infinite domain can mainly be solved efficiently on a finite computational domain. It has been applied successfully for classical integer-order elliptic, parabolic and other famous equations, such as Poisson equations [50], heat equations [51–54], Schrödinger equations [55–59], Navier–Stokes equations [60], parabolic Volterra integro-differential equations [61,62] and some others [63,64]. About the relevant details of ABMs, the readers can further refer to the review papers [65,66] and references therein. Until now, we are not aware of any published work on numerical solutions for FPDEs on unbounded domains. Thus, we dedicate this work to investigate the problem by using artificial boundary method and to fill the gap as an attempt.

In the present work, Section 2 devotes to the reduction of the original problem into one on a finite domain by using the exact artificial boundary condition. Based on the achievements of Section 2, an effective finite difference scheme (3.18)–(3.21) is presented for the reduced problem in Section 3, where the method of order reduction [23,52,67] is used. In the following section, we shall focus our attention on the stability and convergence analysis of finite difference scheme by using the discrete energy method. Two numerical experiments are carried out in Section 5 to verify the numerical efficiency and accuracy. A brief conclusion ends the paper.

2. Reduction of the problem

We discuss a class of fractional differential equations on an unbounded domain expressed by