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Multilevel schemes for the shallow water equations

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Abstract

In this paper, we study a number of multilevel schemes for the numerical solution of the shallow water equations; new schemes and new perspectives of known schemes are examined. We consider the case of periodic boundary conditions. Spatial discretization is obtained using a Fourier spectral Galerkin method. For the time integration, two strategies are studied. The first one is based on scale separation, and we choose the time scheme (explicit or semi-implicit) as a function of the spatial scales (multilevel schemes). The second approach is based on a splitting of the operators, and we choose the time integration method as a function of the operator considered (multistep or fractional schemes). The numerical results obtained are compared with the explicit reference scheme (Leap–Frog scheme), and with the semi-implicit scheme (Leap–Frog scheme with Crank–Nicholson scheme for the gravity terms), both computed with a similar mesh. The drawback of the explicit reference scheme being the numerical stability constraint on the time step, and the drawback of the semi-implicit scheme being the dispersive error, the aim with the new schemes is to obtain schemes with less dispersive error than the semi-implicit scheme, and with better stability properties than the explicit reference scheme. The numerical results obtained show that the schemes proposed allow one to reduce the dispersive error and to increase the numerical stability at reduced cost.

Keywords: Multilevel methods; Shallow water equations; Fractional step methods; Inertial waves; Gravity waves

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1. Introduction: motivation of the problem

We consider the two-dimensional nonlinear shallow water problem, with periodic boundary conditions (doubly periodic f-plane). This problem is considered as a planetary model for the simulation of atmospheric or oceanic flows. The equations are written as follows:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{\Omega} \times \mathbf{u}) + f\mathbf{u}^{\perp} + \nabla\left(gh + \frac{1}{2}|\mathbf{u}|^2\right) = 0, \\ &\frac{\partial h}{\partial t} + \operatorname{div}((H+h)\mathbf{u}) = 0, \end{aligned} \tag{1.1}$$

where $\mathbf{u} = (u, v)^{\top}$ is the velocity field, $\mathbf{u}^{\perp} = (-v, u)^{\top}$ the orthogonal velocity field, $\mathbf{\Omega} = \nabla \times \mathbf{u}$ the vorticity vector, h the height of the free surface around H, and $|-|$ the Euclidean norm.

Here, we have considered the formulation of the shallow water problem with the following scalar dependent variables instead of the velocity vector $\mathbf{u}$: the *vorticity* $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and the *plane divergence* $\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$. So, the problem considered is as follows:

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \frac{\partial(\omega u)}{\partial x} + \frac{\partial(\omega v)}{\partial y} + f\delta = 0, \\ &\frac{\partial \delta}{\partial t} + \frac{\partial(\omega u)}{\partial y} - \frac{\partial(\omega v)}{\partial x} - f\omega + \Delta\left(gh + \frac{1}{2}|\mathbf{u}|^2\right) = 0, \\ &\frac{\partial h}{\partial t} + H\delta + \operatorname{div}(h\mathbf{u}) = 0. \end{aligned} \tag{1.2}$$

It is necessary to supplement these equations with initial conditions for ω, δ, h, and boundary conditions (periodicity in the two directions). The computational domain Ω considered is $\Omega = (0, L_x) \times (0, L_y)$, with $L_x = L_y = 6.31 \times 10^6$ m (earth radius), the period in the x and y directions.

The problem (1.2) induces substantial numerical difficulties if we want to compute directly the numerical approximation of (1.2). Indeed, most atmospheric flows are turbulent flows, i.e., they contain a wide range of scales with very different spatial size and characteristic times. To overcome these numerical difficulties, large eddy simulation (LES) models for turbulence modeling are usually proposed, in order to compute only the large scales of the flow (which contain most of the kinetic energy and the enstrophy in two-dimensional turbulent flows), and modeling the dissipative action of the small scales (which are not computed) on the large ones (see, for example [36,39] for more details). In meteorology, a model often used consists in adding a hyperdissipative operator in Eqs. (1.2) (see [3,5,6,21,35,44]). Such an operator is of the form:

$$\nu_{\mathrm{T}}\Delta^{2p} \tag{1.3}$$

with p an integer parameter and ν_{T} the turbulent viscosity (or eddy viscosity):

$$\nu_{\mathrm{T}} = \frac{\zeta}{k'^{4p}_{\max}\Delta t}. \tag{1.4}$$

Here, Δt is the time step retained for the numerical computation, $k'_{\max}$ is the modulus of the highest wavenumber associated with the smallest computed scales: $k'_{\max} = \sqrt{2}\frac{2\pi}{L_x}(\frac{N}{2}) = \sqrt{2}\frac{2\pi}{L_y}(\frac{N}{2})$, ζ is a nondimensional positive constant and p is an integer as in (1.3). In practice, for the numerical simulations described in this paper, we have chosen $p = 2$ and $\zeta = 10^4$.

As we will see in the next section, the role of the additive term (1.3) in Eqs. (1.2) is to prevent spectral reflections in the high wavenumbers of the spectra, in order to obtain an energy spectrum (velocity spectrum) with a slope of k'^{-3} in the inertial range, in agreement with the two-dimensional homogeneous