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Journal of Computational Physics 207 (2005) 129–150
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A nodal triangle-based spectral element method for the shallow water equations on the sphere

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Received 17 August 2004; received in revised form 11 January 2005; accepted 12 January 2005
Available online 16 February 2005

Abstract

A nodal triangle-based spectral element (SE) method for the shallow water equations on the sphere is presented. The original SE method uses quadrilateral elements and high-order nodal Lagrange polynomials, constructed from a tensor-product of the Legendre–Gauss–Lobatto points. In this work, we construct the high-order Lagrange polynomials directly on the triangle using nodal sets obtained from the electrostatics principle [J.S. Hesthaven, From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex, SIAM Journal on Numerical Analysis 35 (1998) 655–676] and Fekete points [M.A. Taylor, B.A. Wingate, R.E. Vincent, An algorithm for computing Fekete points in the triangle, SIAM Journal on Numerical Analysis 38 (2000) 1707–1720]. These points have good approximation properties and far better Lebesgue constants than any other nodal set derived for the triangle. By employing triangular elements as the basic building-blocks of the SE method and the Cartesian coordinate form of the equations, we can use any grid imaginable including adaptive unstructured grids. Results for six test cases are presented to confirm the accuracy and stability of the method. The results show that the triangle-based SE method yields the expected exponential convergence and that it can be more accurate than the quadrilateral-based SE method even while using 30–60% fewer grid points especially when adaptive grids are used to align the grid with the flow direction. However, at the moment, the quadrilateral-based SE method is twice as fast as the triangle-based SE method because the latter does not yield a diagonal mass matrix.

Keywords: Dubiner; Electrostatics; Fekete; Finite element; Icosahedron; Jacobi; Koornwinder; Lagrange; Platonic solids; Polynomial; Proriol; Quadrilateral; Spherical geometry

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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2005.01.004

1. Introduction

The recent trend towards distributed-memory computers having thousands of commodity processors has rekindled interest in the development of local high-order methods for the simulation of geophysical fluid dynamics applications. The most common local high-order method is the spectral element (SE) method. The SE method can be constructed in modal (spectral) or nodal (physical) space. In addition, the building-blocks (or element shapes) used for the SE method have been the quadrilateral or the triangle. If quadrilaterals are used then the SE method is typically employed in nodal space where the basis functions are constructed from a tensor-product of the one-dimensional Legendre cardinal functions [16]. However, if triangles are used then the SE method has been typically employed only in modal space [26]. This was due to the lack of a good set of nodal points for the triangle.

Using the electrostatics principle, Hesthaven [17] obtained a set of nodal points on the triangle with good approximation properties for all polynomials of order $N < 11$. For $11 \leqslant N \leqslant 15$, we use the Fekete points [29] which are only currently available for orders $N < 20$. Using this nodal set, Warburton et al. [31] showed exponential convergence for the incompressible Navier–Stokes equations. The success of these results has inspired us to seek similarly successful applications of this nodal triangular set for the solution of the shallow water equations on the sphere.

The shallow water equations are a set of first-order nonlinear hyperbolic equations, which contain all of the horizontal operators found in the primitive atmospheric equations used in numerical weather prediction (NWP) and climate models. Thus to design a good atmospheric model requires a good shallow water model. The construction of fast, accurate, and flexible atmospheric models is the ultimate goal of our research. In this quest, we have successfully developed an exponentially convergent global atmospheric model using the nodal quadrilateral-based SE method [13,14]. While the accuracy and performance of this model have been shown to exceed those of operational spectral transform models, developing adaptive grids for quadrilateral elements may prove too cumbersome to pursue. The existence of numerous adaptive triangular mesh generation packages [1,8] motivates us to explore nodal triangle-based SE methods.

The rest of the paper is organized as follows. Section 2 describes the governing equations of motion used to test our numerical method. In Section 3, we describe the discretization of the governing equations. This includes the spatial discretization by the triangle-based SE method and the time-integrator. In Section 4, we describe a few of the many possible triangular tessellations of the sphere. Finally, in Section 5, we present convergence rates of the nodal triangle-based SE method and compare it with the quadrilateral-based SE method. This then leads to some conclusions about the feasibility of this approach for constructing future atmospheric models and a discussion on the direction of future work.

2. Shallow water equations

The shallow water equations are a system of first-order nonlinear hyperbolic equations, which govern the motion of an inviscid incompressible fluid in a shallow depth. The predominant feature of this type of fluid is that the characteristic length of the fluid is far greater than its depth which is analogous to the motion of air in the atmosphere and water in the oceans. For this reason, these equations are typically used as a first step toward the construction of either NWP, climate, or ocean models.

The spherical shallow water equations in Cartesian advective form are

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \boldsymbol{u}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -f(\boldsymbol{x} \times \boldsymbol{u}) - \nabla(\phi + \phi^{s}) - \mu \boldsymbol{x}, \tag{2}$$