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A sharp interface finite volume method for elliptic equations on Cartesian grids

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ABSTRACT

We present a second order sharp interface finite volume method for the solution of the three-dimensional elliptic equation $\nabla \cdot (\beta(\vec{x})\nabla u(\vec{x})) = f(\vec{x})$ with variable coefficients on Cartesian grids. In particular, we focus on interface problems with discontinuities in the coefficient, the source term, the solution, and the fluxes across the interface. The method uses standard piecewise trilinear finite elements for normal cells and a double piecewise trilinear ansatz for the solution on cells intersected by the interface resulting always in a compact 27-point stencil. Singularities associated with vanishing partial volumes of intersected grid cells are removed by a two-term asymptotic approach. In contrast to the 2D method presented by two of the authors in [M. Oevermann, R. Klein, A Cartesian grid finite volume method for elliptic equations with variable coefficients and embedded interfaces, Journal of Computational Physics 219 (2006) 749–769] we use a minimization technique to determine the unknown coefficients of the double trilinear ansatz. This simplifies the treatment of the different cut-cell types and avoids additional special operations for degenerated interface topologies. The resulting set of linear equations has been solved with a BiCGSTAB solver preconditioned with an algebraic multigrid. In various testcases – including large β-ratios and non-smooth interfaces – the method achieves second order of accuracy in the L_∞ and L_2 norm.

1. Introduction

We seek solutions of the three-dimensional variable coefficient elliptic equation

$$\nabla \cdot (\beta(\vec{x})\nabla u(\vec{x})) = f(\vec{x}), \quad \vec{x} \in \overline{\Omega} \setminus \Gamma \tag{1}$$

defined in a domain $\overline{\Omega} \setminus \Gamma$ with an embedded interface Γ. For simplicity we assume $\overline{\Omega}$ to be a simple cuboid. The embedded interface Γ separates two disjoint sub-domains $\overline{\Omega}^+$ and $\overline{\Omega}^-$ with $\overline{\Omega} = (\overline{\Omega}^+ \cup \overline{\Omega}^-)$, see Fig. 1 for an illustration. Along the interface we prescribe jump conditions for the solution

$$[\![u]\!]_\Gamma = u^+(\vec{x}) - u^-(\vec{x}) = g(\vec{x}_\Gamma) \tag{2}$$

and for its gradient in the normal direction

$$[\![\beta u_n]\!]_\Gamma = \beta^+ u_n^+ - \beta^- u_n^- = h(\vec{x}_\Gamma), \tag{3}$$

with the notation $u_n = (\nabla u \cdot \vec{n})$. The unit normal vector $\vec{n}$ on Γ is defined to point from $\overline{\Omega}^+$ to $\overline{\Omega}^-$.

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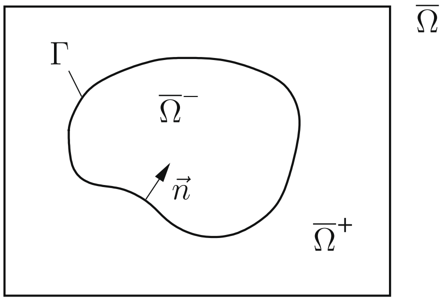

Fig. 1. Domain $\overline{\Omega}$ with sub-domains $\overline{\Omega}^+$, $\overline{\Omega}^-$, and embedded interface Γ.

Elliptic equations of type (1) with variable and discontinuous coefficients and solution discontinuities often arise as a component in modelling physical problems with embedded boundaries. Examples include incompressible two-phase flow with surface tension featuring jumps in pressure and pressure gradient across the interface, projection methods for zero Mach-number premixed combustion with jumps in the dynamic pressure and pressure gradient across the flame front, heat conduction between materials of different heat capacity and conductivity and interface diffusion processes. In the literature one can find a vast number of different approaches for the numerical solution of this type of problem. However, we limit our discussion here to methods on grids which are not aligned with the interface. These methods have the advantage that they do not need any re-meshing if the interface moves.

In Peskin's immersed boundary method [31], singular forces arising from discontinuous coefficients and jump conditions are treated as delta functions. Using discretised discrete delta functions, the discontinuity is spread over several grid cells making the method first order accurate. The method has been used for many problems in mathematical biology and fluid mechanics. Cortez and Minion [3] considerably improved Peskin's method by improving its accuracy through higher order procedures for representing boundary forces. Recent work by Tornberg and Engquist [38,39,5] generalizes the immersed boundary approach and allows for high order approximations with minimal distribution of discontinuities or singular source terms over the computational grid.

Mayo [25,26] presented a second order accurate method for Poisson's equation and the biharmonic equation on irregular domains using an integral equation formulation. The resulting Fredholm integral equations of the second kind are solved with a fast Poisson solver on a rectangular region. Although the method captures solution discontinuities at the embedded interface, continuous derivatives have been assumed to evaluate the discrete Laplacian. The method can easily be extended to fourth order accuracy.

The immersed interface method [16–18,20] is a second order finite difference method on Cartesian grids for second order elliptic and parabolic equations with variable coefficients. Discontinuities in the solution and the normal gradient at the interface are explicitly incorporated into the finite difference stencil. Second order has been achieved by including additional points near the interface into the standard 5-point stencil leading to a non-standard six-point stencil in 2D. The resulting linear equation system is sparse but not symmetric or positive definite. Based on the immersed interface method Li and Ito [19] present a second order finite difference method which satisfies the sign property on the matrix coefficients which guarantees the discrete maximum principle. The resulting linear system of equations is non-symmetric but diagonally dominant and its symmetric part is negative definite. The ideas presented in [19] have been extended to 3D in [4].

A first order finite difference method on Cartesian grids was presented by Liu et al. [22]. Interface jump conditions are explicitly incorporated into the finite difference stencil as in the immersed interface method. Applying a one-dimensional approach in each spatial direction by implicitly smearing out the gradient jump condition, standard stencils (5-point in 2D, 9-point in 3D) for the discrete Laplacian are achieved leading to a symmetric positive definite matrix for the Poisson equation. The method shows first order accuracy for the solution u in the L^∞-norm for constant coefficients $\beta^\pm$. A convergence proof of the method has been provided in [23] based on the weak formulation of the problem. Due to its simplicity and robustness the methods has been used in many engineering and scientific problems. The method has been independently developed and applied to incompressible two-phase flow in [29].

A fourth order accurate finite difference method for elliptic problems with complex boundaries has been developed by Gibou and Fedkiw in [7]. By high order extrapolation of the solution outside the domain they were able to apply high order finite difference formulas at and near the interface. Similar ideas have been used in a series of papers by Wei and coworkers [44,43,42] for elliptic problems with embedded interfaces. They developed finite difference methods of up to sixth order in 3D for smooth interfaces and up to second order for complex interfaces with sharp edges, wedges, and tips. Their methods can be viewed as a higher-order generalization of the immersed interface method. Solutions on both sides of the interface are smoothly extended beyond the interface allowing the application of standard high order finite difference formulas.

One of the first methods to model discontinuities in the finite element framework without aligning the grid with the interface has been presented in [27,1]. In the so-called extended finite element method the original finite element space is enriched by additional basis functions introducing new unknowns to the problems. The choice of additional enrichment functions depends on the type of discontinuity, e.g. step functions for solution discontinuities or distance functions for kinks