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Anti-diffusive flux corrections for high order finite difference WENO schemes

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Abstract

In this paper, we generalize a technique of anti-diffusive flux corrections, recently introduced by Després and Lagoutière [Journal of Scientific Computing 16 (2001) 479–524] for first-order schemes, to high order finite difference weighted essentially non-oscillatory (WENO) schemes. The objective is to obtain sharp resolution for contact discontinuities, close to the quality of discrete traveling waves which do not smear progressively for longer time, while maintaining high order accuracy in smooth regions and non-oscillatory property for discontinuities. Numerical examples for one and two space dimensional scalar problems and systems demonstrate the good quality of this flux correction. High order accuracy is maintained and contact discontinuities are sharpened significantly compared with the original WENO schemes on the same meshes.

Keywords: Anti-diffusive flux correction; Sharpening contact discontinuity; High order accuracy; WENO scheme

1. Introduction

In this paper, we are interested in improving the high order finite difference weighted essentially non-oscillatory (WENO) schemes in the resolution of contact discontinuities. We use the fifth-order finite difference WENO scheme in [11] as an example to demonstrate our approach, although the technique can also be used on finite difference WENO schemes of other orders of accuracy [1] and finite volume WENO schemes on rectangular and triangular meshes [10,13,16].

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High order finite difference WENO schemes in [11] were developed based on the successful ENO schemes [9,20,21] and third-order finite volume WENO schemes [13], and have been quite successful in computational fluid dynamics and other applications. They are especially suitable for problems containing both shocks and complicated smooth flow features. For more details, we refer to the lecture notes [18] and the survey paper [19], and the references therein.

However, as a disadvantage of all shock capturing schemes, the high order finite difference WENO schemes in [11] also suffer from a relatively poor resolution of contact discontinuities, comparing with the resolution of shocks. Although in general a higher order method does have sharper contact discontinuity resolution than lower order methods, and the resolution can often be improved by using a better Riemann solver or flux splitting, it is still in general not easy to stop the progressively more severe smearing for longer simulation time, and at the same time to maintain non-oscillatory property and high order accuracy of the scheme. This is particularly challenging in two or more spatial dimensions.

There have been a lot of efforts in the literature to overcome this problem. Harten [7] proposed the artificial compression method, which modifies the numerical flux so that the numerical characteristics, instead of being in parallel or diverging from the contact discontinuity, converge slightly towards the contact discontinuity to keep its sharpness. Of course, this compression must be done carefully so that stability and accuracy are not lost, and smooth parts of the solution are not evolved to small staircases. This idea was later generalized by Yang [28] to higher order finite volume ENO schemes, via a slope modification. The approach of Yang [28] maintains higher order accuracy of the original ENO scheme while significantly sharpens contact discontinuities. Yang's approach was also applied to finite difference ENO schemes [21] and WENO schemes [1,11] with equally good results. However, two-dimensional results using this approach are less satisfactory. Another successful strategy is Harten's subcell resolution idea [8], which uses two pieces of different polynomial reconstructions in the discontinuity cell, instead of the usual single polynomial reconstruction in a cell, in order not to smear the discontinuity. This strategy works well for one-dimensional problems [8,21]. However, it seems to be difficult to generalize this idea to multi-dimensional problems, except for some special situations, e.g. [22].

More recently, Després and Lagoutière [3] proposed a new approach called limited downwind scheme, much akin to a class of flux limiters by Sweby [24], to prevent the smearing of contact discontinuities while keeping non-linear stability. Their scheme is identical with the ultrabee scheme developed by Roe [15] in the case of linear advection. By introducing an anti-diffusive flux, it gives remarkably sharp profile of contact discontinuities in both one-dimensional scalar and system cases. More importantly, they observe numerically and prove theoretically that their scheme adopts a class of *moving* traveling wave solutions exactly. This has an important implication that the smearing of contact discontinuities will not be progressively more severe for larger time, but will be stabilized for all time. A later paper by Bouchut [2] further modifies this scheme to satisfy entropy conditions and also gives a simple explicit formula for this limited downwind anti-diffusive flux.

Even though the schemes in [2,3] are quite attractive, they are only first-order accurate and are not suitable for computing solutions containing both discontinuities and complex smooth flow features. Our objective in this paper is to develop an anti-diffusive flux correction technique, based on the approach of [2,3], to high order finite difference WENO schemes in [1,11]. We would like the resulting scheme to maintain high order accuracy in smooth regions, non-oscillatory behavior near discontinuities, and sharp contact discontinuity resolution similar to the first-order schemes in [2,3] which does not progressively become worse for larger time.

The first-order schemes developed in [2,3] use simple Euler forward time discretization. For our high order finite difference schemes, we will need to use high order total variation diminishing (TVD) Runge–Kutta time discretizations [20]. It is more difficult to maintain sharp contact discontinuity resolution with multistage high order Runge–Kutta methods. We will need to introduce adjustments in each stage of the Runge–Kutta methods in order to overcome this difficulty.