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A Chebyshev/rational Chebyshev spectral method for the Helmholtz equation in a sector on the surface of a sphere: defeating corner singularities

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Abstract

When the boundaries of a domain meet at an angle, the solutions to an elliptic partial differential equation will usually be singular at the corner. Using the example of the Helmholtz equation on the surface of a sphere in a domain bounded by meridians, we show how corner singularities can be defeated by mapping the corner to infinity. By applying a Chebyshev series in longitude and a rational Chebyshev series in the "Mercator" coordinate, $y = \operatorname{arctan h}(\cos(\text{colatitude}))$, we obtain an exponential rate of convergence despite the corner singularities.

Keywords: Pseudospectral; Chebyshev polynomial; Rational Chebyshev functions; Corner singularities

1. Introduction

In spherical coordinates (λ, μ), where λ is longitude and μ is the cosine of colatitude, the Helmholtz equation on the surface of a sphere is

$$\frac{1}{1-\mu^2}u_{\lambda\lambda} + (1-\mu^2)u_{\mu\mu} - 2\mu u_{\mu} + k^2 u = f(\lambda,\mu), \tag{1}$$

where k is a constant. The domain is the sector bounded by the meridians $\lambda = 0$, Ξ, where Ξ is a constant as shown in Fig. 1. Similar sectorial wave problems arise in ocean tides [7,10,19,21] , but for expository

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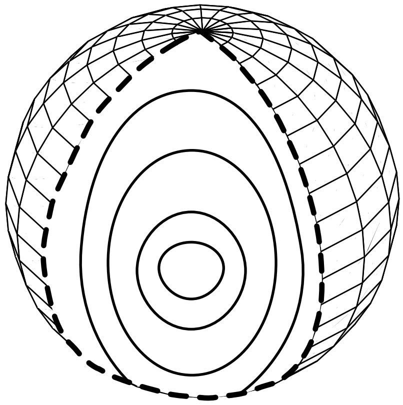

Fig. 1. Schematic of a sectorial domain on the surface of a sphere. The thick dashed lines are the boundaries of the sector, defined by two meridians, $\lambda = 0$ and $\lambda = \Xi$, where λ is longitude. The thick solid curves are the schematic isolines of a typical solution. Note that the two meridians meet at an angle, forming a corner at the north pole; a similar corner exists at the south pole (not visible).

purposes the simpler Helmholtz equation is better. For simplicity, we shall discuss only homogeneous Dirichlet boundary conditions in most of the article, but we shall explain the easy generalization to inhomogeneous boundary conditions in Section 6.

2. Corner singularities

Corner singularities are the theme of books by Grisvard [11] and Kozlow, Mazya, and Rossman [15] and Kondratiev's long review [14]; the numerical implications and various remedies have been discussed by many authors including [2,3,5,9,13,17,20,24,25]. For the present problem, note that (1) (for $k = 0$) has homogeneous solutions

$$\sin(s\lambda)P_n^s(\mu), \quad s = j(\pi/\Xi), \quad j = 1, 2, \ldots \tag{2}$$

It is well-known [1] that the associated Legendre function $P_n^s(\mu)$ is singular at both poles as $(1-\mu^2)^{s/2}$. The homogeneous and particular solutions for the Helmholtz equations are generally singular, too.

3. Mercator coordinate

Almost half a millenia ago, the prolific cartographer Gerhard Mercator introduced the stretched latitudinal coordinate that bears his name:

$$\mu = \tanh(y) \leftrightarrow y = \operatorname{arctanh}(\mu). \tag{3}$$

This transformation is useful because

$$(1 - \mu^2)^{s/2} = \operatorname{sech}^s(y). \tag{4}$$

The branch points at $\mu = \pm 1$ have been moved to infinity. As explained in [5], the hyperbolic secant function, raised to any power, decays exponentially as $|y| \to \infty$. Any reasonable basis set for the infinite interval will yield a spectral series that converges *exponentially fast* [8,5].