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A fast, higher-order solver for scattering by penetrable bodies in three dimensions

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Abstract

In this paper, we introduce a new fast, higher-order solver for scattering by inhomogeneous media in three dimensions. As in previously existing methods, the low complexity of our integral equation method, $\mathcal{O}(N\log N)$ operations for an N point discretization, is obtained through extensive use of the fast Fourier transform (FFT) for the evaluation of convolutions. However, the present approach obtains significantly higher-order accuracy than these previous approaches, yielding, at worst, third-order far field accuracy (or substantially better for smooth scatterers), even for discontinuous and complex refractive index distributions (possibly containing severe geometric singularities such as corners and cusps). The increased order of convergence of our method results from (i) a partition of unity decomposition of the Green's function into a smooth part with unbounded support and a singular part with compact support, and (ii) replacement of the (possibly discontinuous) scatterer by an appropriate "Fourier smoothed" scatterer; the resulting convolutions can then be computed with higher-order accuracy by means of $\mathcal{O}(N\log N)$ FFTs. We present a parallel implementation of our approach, and demonstrate the method's efficiency and accuracy through a variety of computational examples. For a very large scatterer considered earlier in the literature (with a volume of $3648\lambda^3$, where λ is the wavelength), using the same number of points per wavelength and in computing times comparable to those required by the previous approach, the present algorithm produces far-field values whose errors are two orders of magnitude smaller than those reported previously.

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1. Introduction

Scattering problems find application in a wide range of fields including communications, materials science, plasma physics, biology, medicine, radar, and remote sensing. The evaluation of useful numerical solutions for scattering problems remains a highly challenging problem requiring novel mathematical approaches and powerful computational tools: applications of interest make it necessary to evaluate highly oscillatory fields in large, complex three-dimensional geometries. Hence, without an extremely efficient and high-order accurate method, the computation of solutions to such problems is infeasible even with modern computing hardware. In this paper, we present a new fast, higher-order method for evaluation of scattering by inhomogeneous media in three dimensions, which in $\mathcal{O}(N \log N)$ operations (where N is the total number of discretization points) is able to produce highly accurate solutions for problems of unprecedented size. Before describing our method, we formulate the problem and present an overview of previous work in this area.

Given an incident field u^{i}, we denote by u the total field – which equals the sum of u^{i} and the resulting scattered field u^{s}:

$$u = u^{\mathrm{i}} + u^{\mathrm{s}}. \tag{1}$$

Calling λ the wavelength of the incident field and $\kappa = \frac{2\pi}{\lambda}$ the corresponding wavenumber, we require that the total field u satisfies [1, p. 2]

$$\Delta u + \kappa^2 n^2(x) u = 0, \quad x \in \mathbb{R}^3, \tag{2}$$

where the given incident field u^{i} is assumed to satisfy

$$\Delta u^{\mathrm{i}} + \kappa^2 u^{\mathrm{i}} = 0, \quad x \in \mathbb{R}^3. \tag{3}$$

Finally, to guarantee that the scattered wave is outgoing, u^{s} is required to satisfy the Sommerfeld radiation condition [1, p. 67]

$$\lim_{r\to\infty} r\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - \mathrm{i}\kappa u^{\mathrm{s}}\right) = 0. \tag{4}$$

The algorithms available for computing solutions to this problem fall into two broad classes: (i) finite element and finite difference methods and (ii) integral equation methods. Use of finite element and finite difference methods can be advantageous in that, unlike other methods, they lead to sparse linear systems. Their primary disadvantage, on the other hand, lies in the fact that in order to satisfy the Sommerfeld radiation condition (4), a relatively large computational domain containing the scatterer must be used, together with appropriate absorbing boundary conditions on the boundary of the computational domain (see, for example, [2–7]). Thus, these procedures give rise to very large numbers of unknowns and, hence, to very large linear systems. In addition, accurate absorbing boundary conditions with efficient numerical implementations are quite difficult to construct: the error associated with such boundary conditions typically dominates the error in the computed solution.

A second class of algorithms is based on the use of integral equations. An appropriate integral formulation for our problem is given by the Lippmann–Schwinger equation [1, p. 214]

$$u(x) = u^{\mathrm{i}}(x) - \kappa^2 \int_{\Omega} g(x-y) m(y) u(y) \mathrm{d}y, \tag{5}$$