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The Shapley–Shubik power index for games with several levels of approval in the input and output

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Abstract

Voting systems with several levels of approval in the input and output are considered in this paper. That means games with $n \geq 2$ players, $j \geq 2$ ordered qualitative alternatives in the input level and $k \geq 2$ possible ordered quantitative alternatives in the output. We introduce the Shapley–Shubik power index notion when passing from ordinary simple games or ternary voting games with abstention to this wider class of voting systems. The pivotal role of players is analysed by means of several examples and an axiomatization in the spirit of Shapley and Dubey is given for the proposed power index.

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1. Introduction, summary and background

The use of game theory to study the distribution of power in voting systems can be traced back to the invention of simple games by Von Neumann and Morgenstern [19] in their 1944 classic, *Theory of Games and Economic Behavior*. The definition of simple games covers most of the familiar examples of constitutional political machinery, among them weighted voting, direct majority rule, relative majority rule, bicameral or multicameral legislatures, veto situations, etc.

Several approaches yield indices which can be interpreted directly in terms of the a priori ability of the players to affect the outcome. The two most conspicuous representatives of this line of research are the Shapley–Shubik power index [8,17,18] and the Banzhaf–Coleman power index [2,7]. A wide collection of studies providing different axiomatizations and other power indices notions has been developed since then by several scientists.

In practice when considering voting systems it is observed that abstention plays a key role in many of the real voting systems that have been modelled by these games (such as the United Nations Security Council, or the United States federal system), yet simple games, by their very nature, cannot take the possibility of abstention into account; those who do not vote 'yes' are presumed to vote 'no'. Indeed, Felsenthal and Machover ([11], p. 22) have remarked on the extent to which some authors 'misreport the rules as though abstention were not a distinct option' and offer the hypothesis that 'the misreporting is due

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to what philosophers of science have called theory-laden or theory-biased observation—a common occurrence, akin to optical illusion, whereby an observer's perception is unconsciously distorted so as to fit a preconception' ([11], p. 280, as well as [10,12]).

One factor that may have hindered the study of games with multiple levels of approval is the absence of a completely satisfactory definition of weighted voting in this context. Indeed, for (j, k) simple games introduced by Freixas and Zwicker [14], a natural weighted notion is proposed and a combinatorial characterization is given in terms of 'grade trade robustness' for weighted (j, k) games within the class of all (j, k) simple games. The framework models voting systems which meet the following conditions: (a) several levels of approval are permitted in the input, say j, (b) several levels of approval are permitted in the output, say k, and (c) those levels are qualitatively ordered.

Here we intend to provide an a priori Shapley–Shubik (S–S) power index for (j, k) simple games. In these games, each individual voter expresses one of j possible levels of input support, and the output consists of one of k possible levels of collective support. Standard simple games are (2, 2) simple games, (3, 2) simple games allow each voter a middle option, which may be interpreted as 'I abstain.' In a seminal work by Felsenthal and Machover ([11], pp. 291–293) it is proposed a Shapley–Shubik power index notion for ternary voting systems (our (3, 2) simple games with abstention). Here we will extend their approach to (j, k) simple games, a topic being its derivation from axioms closely related to Dubey and Shapley's axiomatization [9] for simple games.

Revising the literature on the several attempts to generalize simple games we find that the most relevant examples of (j, k) simple games are those were abstention plays a key role. Several real voting systems have been modelled by these games, such as the United Nations Security Council, or the United States federal system. An important and isolated earlier work on abstention can be found in Fishburn ([13], pp. 53–55). Fishburn's context can be viewed as a special case of our (j, k) simple games (in which $j=3=k$, the game is constant-sum, and the intermediate output level of approval is only achieved when the vote is tied exactly). More recently, several works by Felsenthal and Machover [10–12] have been devoted to the study of voting systems with abstention, and outline the rudiments of a theory of a priori voting power with abstention. Their 'ternary voting rules' correspond to our (3, 2) simple games with the three input alternatives: 'yes,' 'abstention' and 'no.' In the alternative model proposed by Braham and Steffen [6], abstention does not really figure expressing an intermediate degree of support between 'yes' and 'no.'

Classical cooperative games have given rise to several generalizations, related to our model, but distinguished in part by the fact that the output of a cooperative game is a cardinal value rather than a discrete level in a finite ordering. Bolger [3–5] deals with the so-called games with n players and r alternatives, in which the r possible inputs are not ordered; each input alternative j attracts its own coalition of supporting voters, and each such coalition is assigned an output cardinal value (so that the total output is an r-tuple of cardinal values). In particular he develops a Shapley value for such class. More recently, Magaña [16], and Amer et al. [1] introduce the closely related r-games and define the Shapley–Shubik index for this type of games. Hsiao and Raghavan [15] consider multi-choice games and defined a Shapley value for that class considering that different actions carry different weights. In their context the inputs are ordered (each agent has an 'effort level'), and the output is a single cardinal value. However, their notion of monotonicity ([15], Definition 2 p. 243) differs from ours.

Let us briefly outline the contents of this paper. Section 2 deals with notation, definitions, the formal description of the class of (j, k) simple games and several examples. Section 3 defines the S–S power index for (j, k) simple games and relates it to an explanatory probability model; the pivotal role of player is also developed. Taking as reference the examples introduced in Section 2, Sections 4 and 5, illustrates how to calculate the S–S power index for the $(j, 2)$ and (j, k) cases, respectively. Finally, Appendix A shows how to derive the S–S power index from a set of axioms for the $(j, 2)$ case. As Felsenthal and Machover points out in their book: 'The theory of voting power in ternary voting systems is in its infancy, and much remains to be discovered'. This paper tries to be a small insight in this research line.