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A generalized Fourier domain: Signal processing framework and applications

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ABSTRACT

In this paper, a signal processing framework in a generalized Fourier domain (GFD) is introduced. In this newly proposed domain, a parametric form of control on the periodic repetitions that occur due to sampling in the reciprocal domain is possible, without the need to increase the sampling rate. This characteristic and the connections of the generalized Fourier transform to analyticity and to the *z*-transform are investigated. Core properties of the generalized discrete Fourier transform (GDFT) such as a weighted circular correlation property and Parseval's relation are derived. We show the benefits of using the novel framework in a spatial-audio application, specifically the simulation of room impulse responses for auralization purposes in e.g. virtual reality systems.

1. Introduction

In this paper we introduce a framework for signal processing in a generalized Fourier domain (GFD). In this domain a special form of control on the periodic repetitions that occur due to sampling in the reciprocal domain is possible, without the need to increase the sampling rate. First in Section 2 we review the definition of the generalized discrete Fourier transform (GDFT) and its associated generalized Poisson summation formula (GPSF), both previously introduced in [1]. Analogous to the periodic extension of a finite-length signal that occurs in standard Fourier theory [2,3], here we introduce the concept of "weighted periodic signal extension" that naturally occurs when working in the GFD. Next we study the connections of the presented theory to spectral sampling, analyticity and the *z*-transform. This analysis also serves as a discussion of the generalized Fourier transform and its relationship to the standard Fourier transform. In Section 3 important properties of the GDFT are derived such as the *weighted* circular correlation property and Parseval's energy relation for the GFD that, together with the previously introduced *weighted* circular convolution theorem for the GDFT, are fundamental to build a general-purpose GFD-based signal processing framework. To finalize our discussion in Section 4 we show how the novel framework can be used in spatial-audio applications such as the simulation of multichannel room impulse responses for auralization purposes in e.g. virtual reality and telegaming systems.

2. A generalized Fourier domain

Let us define the generalized discrete Fourier transform for finite-length signals $x(n)$, $n=\{0,\dots,N-1\}$, with parameter $\alpha \in \mathbb{C}\backslash\{0\}$ as,

$$\mathcal{F}_\alpha\{x(n)\} \triangleq X_\alpha(k) = \sum_{n=0}^{N-1} x(n)e^{\beta n}e^{-j(2\pi/N)kn}, \tag{1}$$

for $k=\{0,\dots,N-1\}$, where $\beta=\log(\alpha)/N$. The inverse GDFT is given by [1],

$$\mathcal{F}_\alpha^{-1}\{X_\alpha(k)\} \triangleq x(n) = \frac{e^{-\beta n}}{N}\sum_{k=0}^{N-1} X_\alpha(k)e^{j(2\pi/N)kn}. \tag{2}$$

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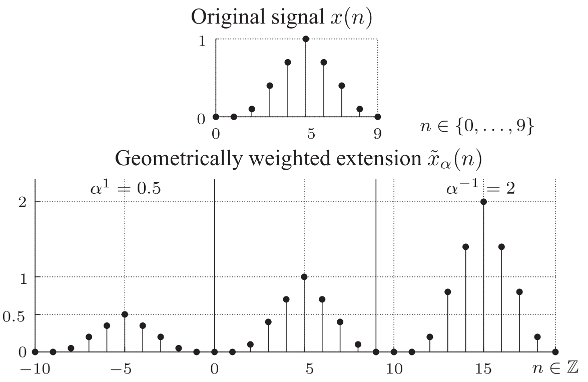

Fig. 1. Geometrically weighted extension of a finite length signal when evaluated outside its original domain, for $\alpha = 0.5$ and $N=10$.

The GDFT (1) is equivalent to the ordinary discrete Fourier transform of the modulated signal $x(n)\mathrm{e}^{\beta n}$. The finite-length signal $\mathrm{e}^{\beta n}$ for $n = \{0,\ldots,N-1\}$, is of finite energy for all $\alpha \in \mathbb{C}\backslash\{0\}$, therefore for $x(n)$ a signal of finite energy, the GDFT can be properly defined [2,1]. Note that when $\alpha = 1$, the transform pair correspond to the standard DFT pair.

Let us denote the periodic extension of $X_\alpha(k)$ by $\tilde{X}_\alpha(k)$ for $k \in \mathbb{Z}$. Clearly we have that $\tilde{X}_\alpha(k) = X_\alpha((k))_N$, where $X_\alpha((k))_N \triangleq X_\alpha(k \bmod N)$, i.e. the circular shift of the sequence is represented as the index modulo N. On the other hand (1) and (2) imply a geometrically *weighted* periodic extension of the signal $x(n)$ when evaluated outside $\{0,\ldots,N-1\}$. This is stated by the generalized Poisson summation formula (GPSF) associated with the transform [1],

$$\tilde{x}_\alpha(n) \triangleq \sum_{p\in\mathbb{Z}} \alpha^p x((n))N = \frac{\mathrm{e}^{-\beta n}}{N}\sum_{k=0}^{N-1} X_\alpha(k)\mathrm{e}^{j(2\pi/N)kn}, \tag{3}$$

where $n \in \mathbb{Z}$, $p = -\lfloor n/N \rfloor$ and $((n))_N = n+pN$. We can regard $\tilde{x}_\alpha(n)$ as a superposition of infinitely many translated and geometrically *weighted* "replicas" of $x(n)$. The replicas outside the support of $x(n)$ are weighted by α^p and $\tilde{x}_\alpha(n) = x(n)$ for $n = \{0,\ldots,N-1\}$. This is illustrated in Fig. 1, where a finite signal and (a part of) its geometrically weighted extension are depicted for $\alpha = 0.5$ and $N=10$. Therefore to work in the generalized Fourier domain implies a manipulation of the signals involved via their geometrically weighted extensions. This is an important fact as we will see through the rest of the paper.

Signals of the form $\tilde{x}_\alpha(n)$ although infinitely long and not being of finite energy can be decomposed into its generalized Fourier transform components by means of (1), evaluating the transform over a signal interval ("period") of length N. This fact follows directly from the generalized Poisson summation formula (3) which shows, that the inverse transform $(\exp(-\beta n)/N)\ \sum_{k=0}^{N-1} X_\alpha(k)\exp(j(2\pi/N)kn)$ is of the form $\tilde{x}_\alpha(n)$ when evaluated over $n = \mathbb{Z}$.

2.1. Connection to sampling, analyticity and the z-transform

The summation formula (3) has an important relationship to spectral sampling. The connection of the (discrete-time) generalized Fourier transform to analyticity and to the z-transform follows as part of the analysis. These relationships are used in a practical application of the theory in Section 4.

Let us begin with the connection to analyticity. Define the standard spectrum of a discrete-time signal by $S(\omega)$, where $\omega \in \mathbb{R}$ represents angular frequency and let $S(\omega) \in L^2[-\pi,\pi]$. Since the original signal $s(n)$, $n \in \mathbb{Z}$, is defined for discrete-time it is clear that $S(\omega)$ is a periodic function of ω with period equal to 2π. The signal $s(n)$ could represent the samples of a continuous-time signal, but without the sampling interval that information is lost and no particular analog representation is to be inferred. Let us assume for a moment that $S(\omega)$ can be analytically continued into the complex angular–frequency plane. This is $S(\omega) \to S(\omega_z)$, where $\omega_z \in \mathbb{C}$ is the complex-valued angular frequency. Let ω_r and ω_i denote the real and imaginary parts respectively of ω_z. Then from the definition of the discrete-time Fourier transform we have,

$$\begin{aligned}
S(\omega_z) &= \sum_{n=-\infty}^{\infty} s(n)\mathrm{e}^{-j\omega_z n},\\
S(\omega_r + j\omega_i) &= \sum_{n=-\infty}^{\infty} s(n)\mathrm{e}^{-j(\omega_r+j\omega_i)n},\\
S(\omega_r + j\omega_i) &= \sum_{n=-\infty}^{\infty} s(n)\mathrm{e}^{\omega_i n}\mathrm{e}^{-j\omega_r n}.
\end{aligned} \tag{4}$$

From here we see that if $s(n)$ is a causal sequence and the analytic continuation of $S(\omega)$ is done on the lower half of the complex plane then $\omega_i < 0$,

$$S(\omega_r + j\omega_i) = \sum_{n=0}^{\infty} s(n)\mathrm{e}^{\omega_i n}\mathrm{e}^{-j\omega_r n},$$

and the extra factor $\mathrm{e}^{\omega_i n}$ can only improve the convergence rate of the series. Now,

$$\begin{aligned}
\lim_{\omega_i \to -0} S(\omega_r + j\omega_i) &= \lim_{\omega_i \to -0} \sum_{n=0}^{\infty} s(n)\mathrm{e}^{-j\omega_r n + \omega_i n},\\
&= \sum_{n=0}^{\infty} \lim_{\omega_i \to -0} s(n)\mathrm{e}^{-j\omega_r n + \omega_i n},\\
&= S(\omega).
\end{aligned}$$

On the other hand we have that

$$\begin{aligned}
&\int_{-\pi}^{\pi} |S(\omega_r + j\omega_i)|^2\, d\omega_r\\
&= \int_{-\pi}^{\pi} S(\omega_r + j\omega_i) S^*(\omega_r + j\omega_i)\, d\omega_r,\\
&= \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} s(n)\mathrm{e}^{-j\omega_r n + \omega_i n} \sum_{m=0}^{\infty} s^*(m)\mathrm{e}^{j\omega_r m + \omega_i m}\right) d\omega_r,\\
&= \sum_{n=0}^{\infty}\left(s(n)\mathrm{e}^{\omega_i n}\sum_{m=0}^{\infty} s^*(m)\mathrm{e}^{\omega_i m}\int_{-\pi}^{\pi}\mathrm{e}^{j\omega_r(m-n)}\, d\omega_r\right),\\
&= 2\pi\sum_{n=0}^{\infty} s(n)\mathrm{e}^{\omega_i n} s^*(n)\mathrm{e}^{\omega_i n},\\
&= 2\pi\sum_{n=0}^{\infty} |s(n)|^2\mathrm{e}^{2\omega_i n} < 2\pi\sum_{n=0}^{\infty}|s(n)|^2,
\end{aligned}$$

where * denotes complex conjugation. Recalling Parseval's relation and noting that $S(\omega) \in L^2[-\pi,\pi]$ implies $s(n) \in l^2(\mathbb{Z})$ [3,2], then for a positive constant C,

$$\int_{-\pi}^{\pi} |S(\omega_r + j\omega_i)|^2\, d\omega_r < C.$$