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TH-collocation for the biharmonic equation

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Abstract

This paper is intended as a contribution to enhance orthogonal collocation methods. In this, a novel collocation method—*TH-collocation*—is applied to the biharmonic equation and the merits of such procedure are exhibited. TH-collocation relaxes the continuity requirements and, for the 2D problems here treated, leads to the development of algorithms for which the matrices are sparse (nine-diagonal), symmetric and positive definite. Due to these properties, the conjugate gradient method can be directly, and more effectively, applied to them. These features contrast with those of the standard orthogonal spline collocation on cubic Hermites, which yields matrices that are non-symmetric and non-positive. This paper is part of a line of research in which a general and unified theory of domain decomposition methods, proposed by Herrera, is being explored. Two kinds of contributions can be distinguished in this; some that are relevant for the parallel computation of continuous models and new discretization procedures for partial differential equations. The present paper belongs to this latter kind of contributions.

Keywords: Trefftz method; Collocation; Domain decomposition; Biharmonic equation; Discontinuous Galerkin

1. Introduction

This paper is part of a line of research in which a general and unified theory of domain decomposition methods (DDM), proposed by Herrera [1] and stemming from Trefftz method [2], is being explored. In it, the terms 'domain decomposition methods' are understood in broader sense than usual and they include many aspects of numerical methods for partial differential equations. As a matter of fact, Herrera's approach to partial differential equations constitutes a general and systematic formulation of discontinuous Galerkin methods [3], in which the use of 'fully' discontinuous functions is permitted. The investigations that are being carried out, in the line of research mentioned above, cover two different aspects. One is concerned with developing novel discretization procedures [4,5] and the other one deals with producing new ways of efficiently using parallel computing resources in the numerical simulation of continuous systems [3].

The main purpose of the present paper is to present an improved orthogonal-collocation treatment of the biharmonic equation. This is based on the application of a new general collocation method, 'TH-collocation', which was introduced in a pair of previous papers [4,5]. An interesting and attractive feature of TH-collocation is the relaxation of the continuity conditions, which allows using trial-spaces of functions that are globally only C^0. This, in turn, permits deriving algorithms with better-structured matrices. In particular, it produces symmetric and positive matrices when it is applied to differential systems with such properties, as is the case of Laplace's and the biharmonic operators. Also, the number of degrees of freedom associated with each node is reduced. For Poisson equation, TH-collocation yields an algorithm of fourth order precision whose global matrix, in addition to being symmetric and positive definite, is block nine-diagonal, with blocks of at most 3×3 [5]. This is to be compared with orthogonal spline collocation (OSC), which for the same order of accuracy yields a global matrix that is neither symmetric, nor positive, and whose blocks are 4×4. Furthermore, TH-collocation also yields another algorithm [5], of second order precision, whose global matrix is strictly nine-diagonal (i.e. with blocks 1×1). Such reduction is not

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possible when OSC is applied. Due to these important advantages, over standard collocation procedures, which TH-collocation possesses, domain decomposition methods (DDM) can be effectively applied to TH-collocation algorithms using the Conjugate Gradient Method (CGM) in a direct manner [3].

For the biharmonic equation semi-analytical discretization procedures, of the Trefftz–Jirousek type [6], have been developed by several authors [7–9] and a review of such methods can be found in [10]. As for non-analytical discretization methods, a recent paper by Lou et al. [11] presents a discussion, and a brief comparison, of several discretization methods that are available to deal with the biharmonic equation. According to them, some of the existing finite difference methods are very efficient and one due to Bjorstad is of optimal complexity. The order of accuracy of such methods is only second order. However, a fourth order collocation algorithm was introduced in [11].

When approaching the discretization of the biharmonic equation with non-analytical procedures, there are mainly two options. The first one consists in using a 13-point stencil [12,13] and in the second one, the 'splitting approach' [13], the biharmonic equation is rewritten as a system of two equations whose treatment requires solving two Poisson equations successively. When this latter procedure is applied, the effectiveness of the method and of its parallel computation depends essentially on those of the Poisson equations. The most popular collocation formulation for partial differential equations of second order, which the majority of the authors working in this field have used up to now, is OSC; i.e. the Hermite bi-cubic orthogonal spline collocation [14]. The OSC formulation is applied in a trial-space of functions which are globally C^1; this produces a global matrix, which in its usual form is neither symmetric nor positive definite, even when the differential operator has these properties.

In this paper, we tackle the biharmonic equation using the splitting approach and solve each one of the Poisson equations by means of TH-collocation, profiting from the advantageous features of the TH-collocation treatment of Poisson equation. Thus far, the order of accuracy of our algorithms has been only derived experimentally, as was done in [5] and in Section 7. However, an interesting characteristic of our method is that it actually produces the same solutions as those obtained by Lou et al. [11]. Using this fact, a rigorous theoretical proof of the fourth order accuracy of our algorithm can be constructed. However, such discussions will be presented elsewhere.

2. Notations

In our formulation the notations $\Omega \subset R^n$ and $\partial\Omega$ are used for a domain of the Euclidean space of dimension n and its boundary, respectively. Throughout this paper n is taken to be equal to 2. Let $\Pi \equiv \{\Omega_1,\ldots,\Omega_E\}$ be a partition of Ω. Given such a partition, the boundaries of the subdomains are $\partial\Omega_i$, $i=1,\ldots,E$. Clearly, $\partial\Omega \subset \cup_{i=1}^{E}\partial\Omega_i$ and the 'internal boundary', Σ, is defined to be the closed complement of $\partial\Omega$ relative to $\cup_{i=1}^{E}\partial\Omega_i$. Then, $\partial\Omega$ will be referred as 'external boundary'. In the external boundary, the unit normal vector is taken pointing outwards. As for the internal boundary, a positive side of Σ and a unit normal vector, also denoted by $\underline{n}$, are defined almost everywhere (a.e.) on it with the convention that $\underline{n}$ points toward the positive side.

It is assumed that for each $i=1,\ldots,E$, there is a linear space $D(\Omega_i)$, whose elements are functions defined in Ω_i. Then, trial and test functions are taken from the linear space D, defined by:

$$D \equiv D(\Omega) \equiv D(\Omega_1) \oplus \cdots \oplus D(\Omega_E) \tag{1}$$

Possible choices for $D(\Omega_i)$ are the Sobolev spaces $H^5(\Omega_i)$, $i=1,\ldots,E$. For the case of elliptic equations of second order that will be considered, it is convenient to take $s \geq 2$. In fact, when the space D is defined by Eq. (1), a function $u \in D$ is a finite sequence of functions $u \equiv (u_1,\ldots,u_E)$ such that $u_i \in D(\Omega_i)$, $i=1,\ldots,E$. It is assumed that the trace of every $u_i \in D(\Omega_i)$ is defined a.e. on $\partial\Omega_i$. Given any function $u \in D$, $u \equiv (u_1,\ldots,u_E)$, two traces are defined at every point of Σ, which are denoted by u_+ and u_-, respectively. Since generally, $u_+ \neq u_-$, it is useful to define the 'jump' and the 'average' of any function $u \in D$ by

$$[u] = u_+ - u_- \quad \text{and} \quad \dot{u} = (u_+ + u_-)/2 \tag{2}$$

respectively. This notation will be applied not only for a function, but for its derivatives as well. Clearly, the definition of the jump of a function is dependent on the orientation of Σ; however, the expressions that will be handled in this paper are invariant with respect to such orientation.

In some previous works, for simplicity, we have written $\mathcal{L}u = f_\Omega$, in Ω, to mean:

$$\mathcal{L}u = f_\Omega, \ \text{ at each } \Omega_i, i = 1, \ldots, E \tag{3}$$

For greater clarity, in the present paper we will be more explicit and write directly, Eq. (3), since $w\mathcal{L}u$ is not, in general, defined on Σ when $u \in D$ and $w \in D$. Similarly, we also write $\sum_{i=1}^{E}\int_{\Omega_i} w\mathcal{L}u \, dx$ instead of $\int_\Omega w\mathcal{L}u \, dx$. Assume a tensor-valued function $\underline{\underline{a}}$ is defined in Ω and write $\underline{a}_n \equiv \underline{\underline{a}} \cdot \underline{n}$, then it can be shown that

$$\sum_{i=1}^{E} \int_{\partial\Omega_i} w\underline{a}_n \cdot \nabla u \, dx = \int_{\partial\Omega} w\underline{a}_n \cdot \nabla u \, dx - \int_{\Sigma} [w\underline{a}_n \cdot \nabla u] dx \tag{4}$$

3. Splitting formulation of the biharmonic equation

The formulation of well-posed problems in function spaces containing discontinuous functions require that some jump of the functions and their derivatives, across the internal boundary, be prescribed. A well-posed boundary