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Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach☆

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ABSTRACT

In this paper, the stabilizability of discrete-time linear switched systems is considered. Several sufficient conditions for stabilizability are proposed in the literature, but no necessary and sufficient. The main contributions are the necessary and sufficient conditions for stabilizability based on the set-theory and the characterization of a universal class of Lyapunov functions. An algorithm for computing the Lyapunov functions and a procedure to design the stabilizing switching control law are provided, based on such conditions. Moreover, a sufficient condition for non-stabilizability for switched system is presented. Several academic examples are given to illustrate the efficiency of the proposed results. In particular, a Lyapunov function is obtained for a system for which the Lyapunov–Metzler condition for stabilizability does not hold.

1. Introduction

Switched systems are systems for which the current time-varying dynamics, specified by the so-called switching law, belongs to a finite set of modes (see Liberzon, 2003). In the last decades, a large literature has been devoted to study switched systems for practical reasons, as they model complex systems like embedded ones, and for theoretical ones, since their behavior and stability properties are neither intuitive nor trivial, (Liberzon & Morse, 1999). Depending on the assumptions on the switching law, different frameworks have risen. The switching law is often considered as a perturbation or as a part of the control inputs.

When the switching law is a perturbation, that is an arbitrary function, sufficient and conservative conditions to ensure the stability have been provided (see for overviews Lin & Antsaklis, 2009; Lunze & Lamnabhi-Lagarrigue, 2009; Sun & Ge, 2011). Several refinements have been proposed to obtain necessary and sufficient conditions for the stability of switched systems. Among these conditions, one can cite the joint spectral radius approach (Bauer, Premaratne, & Durán, 1993; Jungers, 2009; Lin & Antsaklis, 2004); the polyhedral Lyapunov functions (Molchanov & Pyatnitskiy, 1989) and the path-dependent switched Lyapunov ones (Lee & Dullerud, 2007). Specific necessary and sufficient conditions exist for two-dimensional systems (Boscain, 2002) or positive ones (Gurvits, Shorten, & Mason, 2007).

In the case the switching law is a part of the control inputs, sufficient conditions for stabilizability have been provided, mainly by using a *min-switching* policy (Liberzon, 2003, Chapter 3) introduced in Wicks, Peleties, and De Carlo (1994), developed in Kruszewski, Bourdais, and Perruquetti (2011) via BMI and leading to Lyapunov–Metzler inequalities (Geromel & Colaneri, 2006a). Based on the set-induced Lyapunov functions introduced in Blanchini (1995), sufficient conditions for the uniform ultimate boundedness have been proposed for uncertain switched linear systems in Lin and Antsaklis (2003). Nevertheless to the best knowledge of the authors, necessary and sufficient conditions for the stabilizability and for the non-stabilizability of discrete-time switched linear systems do not exist in the literature.

The proposed approach is based on the set-theory and invariance for control and analysis. A seminal work dealing with the characterization of invariance is Bertsekas (1972). More recently, the works Blanchini (1999), Gilbert and Tan (1991) and Kolmanovsky and Gilbert (1998) lay the basis of set-theoretic methods and invariance in control. The relation between contractivity and polyhedral Lyapunov functions is presented in Blanchini (1994).

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In Blanchini (1995) and Molchanov and Pyatnitskiy (1989), it is proved that the existence of a contractive polytope is a necessary and sufficient condition for exponential stability and that polyhedral functions form a universal class of Lyapunov functions for linear parametric uncertain systems. See Blanchini and Miani (2008) for a detailed review on the topic.

This paper provides new necessary and sufficient conditions for stabilizability and sufficient ones for non-stabilizability of discrete-time linear switched systems. A family of nonconvex, homogeneous functions is proved to be a universal class of Lyapunov functions for switched linear systems. The key idea is to use the set-theory offering numerically sound algorithms to check the stabilizability (and providing the stabilizing switching law) or the non-stabilizability. The paper extends the results presented in the preliminary work Fiacchini and Jungers (2013).

The outline of the paper is as follows. In Section 2, preliminaries and set-theory tools are presented. Section 3 is devoted to the new necessary and sufficient conditions for stabilizability. Section 4 provides the stabilizing switching control law. Section 5 presents sufficient conditions for non-stabilizability. Links with the Lyapunov–Metzler approach are discussed in Section 6. The results are illustrated by examples in Section 7, before concluding remarks in Section 8.

Notation: define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \le x \le n\}$ with $n \in \mathbb{N}$. Given $D, E \subseteq \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and $M \in \mathbb{R}^{m\times n}$, define $D + E = \{z = x + y \in \mathbb{R}^n : x \in D,\ y \in E\}$, the scalar multiple $\alpha D = \{\alpha x \in \mathbb{R}^n : x \in D\}$ and $MD = \{Mx \in \mathbb{R}^m : x \in D\}$. Given a set $D \subseteq \mathbb{R}^n$, $\mathrm{int}(D)$ is its interior and ∂D its boundary. The set $\mathbb{B}^n$ is the unit euclidean ball in $\mathbb{R}^n$. The i-th element of a finite set of matrices is denoted as A_i, of a set of sets as Ω^i.

2. Preliminaries

Consider the discrete-time switched system

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law that, at any instant, selects the transition matrix among the finite set $\{A_i\}_{i\in\mathbb{N}_q}$, with $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{N}_q$. Given the initial state x_0 and a switching law $\sigma(\cdot)$, we denote with $x_N^\sigma(x_0)$ the state of the system (1) at time N starting from x_0 by applying the switching law $\sigma(\cdot)$. In some cases σ can be a function of the state, for instance in the case of switching control law, as shown later.

Remark 1. Two main cases have to be discriminated depending on the assumptions on the switching law. If $\sigma(\cdot)$ is supposed to be an arbitrary function of time, that is acting as a perturbation, then the problem of asymptotic stability of the system under every possible switching law is usually considered. For this case the necessary and sufficient conditions for stability exist. When $\sigma(\cdot)$ is considered as a manipulable signal, then the problem of asymptotic stabilizability (simply denoted as stabilizability in what follows) is addressed, that consists in the existence and the characterization of the switching laws that yield asymptotic stability if applied. This is the problem considered in the paper.

A concept widely employed in the context of set-theory and invariance is the C-set, see Blanchini (1995) and Blanchini and Miani (2008). A C-set is a compact, convex set with $0 \in \mathrm{int}(\Omega)$. We define an analogous concept useful for our purpose. For this, we first recall that a set Ω is a star-convex set if there exists $x^0 \in \Omega$ such that every convex combination of x and x^0 belongs to Ω for every $x \in \Omega$.

Definition 1. A set $\Omega \subseteq \mathbb{R}^n$ is a C*-set if it is compact, star-convex with respect to the origin and $0 \in \mathrm{int}(\Omega)$.

We define the analogous of the gauge function of a C*-set as

$$\Psi_\Omega(x) = \min_{\alpha\ge 0}\{\alpha \in \mathbb{R} : x \in \alpha\Omega\}, \tag{2}$$

for the C*-set $\Omega \subseteq \mathbb{R}^n$. In what follows, we will refer to $\Psi_\Omega(x)$ as the Minkowski function of Ω at x, with a slight abuse since it is usually defined for C-sets (or symmetric C-sets), (Blanchini & Miani, 2008; Rockafellar, 1970; Schneider, 1993). Some basic properties of the C*-sets and their Minkowski functions are listed below. The proof is avoided, since they follow directly from the definition.

Property 1. *Any C-set is a C*-set. Given a C*-set $\Omega \subseteq \mathbb{R}^n$, we have that $\alpha\Omega \subseteq \Omega$ for all $\alpha \in [0, 1]$, and the Minkowski function $\Psi_\Omega(\cdot)$ is: homogeneous of degree one, i.e. $\Psi_\Omega(\alpha x) = \alpha\Psi_\Omega(x)$ for all $\alpha \ge 0$ and $x \in \mathbb{R}^n$; positive definite; defined on $\mathbb{R}^n$ and radially unbounded.*

The Minkowski functions induced by C-sets have been used in the literature as Lyapunov functions candidates, see Blanchini (1994). In particular, it has been proved that they provide a universal class of Lyapunov functions for linear parametric uncertain systems, (Blanchini, 1995; Molchanov & Pyatnitskiy, 1989), and switched systems with arbitrary switching, (Lin & Antsaklis, 2009). In this paper, we prove that the Minkowski functions induced by C*-sets form a universal class of Lyapunov functions for switched systems with switching control law. For this, we provide a definition of Lyapunov function for the particular context, in analogy with the definition given in Blanchini (1995) for linear parametric uncertain systems.

Definition 2. A positive definite continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a global Lyapunov function for the system (1) if there exist a positive $N \in \mathbb{N}$ and a switching law $\sigma(\cdot)$, defined on $\mathbb{R}^n$, such that V is non-increasing along the trajectories $x_k^\sigma(x)$ and decreasing after N steps, i.e. $V(x_1^\sigma(x)) \le V(x)$ and $V(x_N^\sigma(x)) < V(x)$, for all $x \in \mathbb{R}^n$.

The Definition 2 is a standard definition of global Lyapunov function (or, better, global control Lyapunov function) except for the N-steps decreasing requirement. On the other hand, such a function implies the convergence of every subsequence in $j \in \mathbb{N}$ of the trajectory, i.e. $x_{i+jN}^\sigma(x)$ for all $i < N$, then also the convergence of the trajectory itself. This, with the stability assured by $V(x_1^\sigma(x)) \le V(x)$, ensures the global asymptotic stabilizability of the switched system.

3. Stabilizability of switched systems

It is proved in Molchanov and Pyatnitskiy (1989) that for an autonomous linear switched system, the origin is asymptotically stable if and only if there exists a polyhedral Lyapunov function, see also Blanchini (1995) and Lin and Antsaklis (2009). Our main objective is to prove that analogous results can be stated in the case that the switching sequence is a properly chosen selection, that is considering it as a control law.

The system (1) is asymptotically stabilizable if there exist a switching law and a Lyapunov function for the resulting time-varying system. The switching law will belong to the class of state-dependent one, that is $\sigma(k) = g(x_k)$, where $g : \mathbb{R}^n \to \mathbb{N}_q$. We define, with a slight abuse of notation, the state-dependent switching law as $\sigma(k) = \sigma(x_k)$.

Assumption 1. The matrices A_i, with $i \in \mathbb{N}_q$, are nonsingular.

Remark 2. Assumption 1 is not restrictive. In fact, the stable eigenvalues of the matrices A_i are beneficial from the stability point of view of the switched systems and poles in zero are related to the most contractive dynamics. Moreover, the results presented in the following can be extended to the general case with appropriate considerations. Finally, recall that sampled linear systems do not present poles in the origin and then real systems satisfy Assumption 1.