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Positivity of discrete singular systems and their stability: An LP-based approach☆

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ABSTRACT

In this paper we present an efficient approach to the analysis of discrete positive singular systems. One of our main objectives is to investigate the problem of characterizing positivity of such systems. Previously, this issue was not completely addressed. We provide easily checkable necessary and sufficient conditions for such problem to be solved. On the other hand, we study the stability of discrete positive singular systems. Note that this is not a trivial problem since the set of admissible initial conditions is not the whole space but it is represented by a special cone. All the conditions we provide are necessary and sufficient and are based on a reliable computational approach via linear programming.

1. Introduction

Over the past few years, singular systems (also referred to as descriptor systems, semi-state systems, implicit systems, differential–algebraic systems, or generalized state-space systems) have constantly gained a great interest. This kind of systems naturally appears in many practical areas such as robotics, compartmental systems, circuit systems, Leontief dynamic models, etc.; see Jódar and Merello (2010), Kunkel and Mehrmann (2006), Riaza (2008) and Silva and de Lima (2003). Their solutions and fundamental properties such as stability and controllability have been fully studied. Important developments took place in the 1980s; see for instance the survey paper (Campbell, 1980; Dai, 1989; Lewis, 1986). In the last three decades, some interesting monographs entirely devoted to many topics of this type of systems have been presented (Campbell, 1980; Dai, 1989; Duan, 2010; Kunkel & Mehrmann, 2006; Lam & Xu, 2006; Riaza, 2008; Virnik, 2008a) along with a vast amount of contributions extending the framework of standard systems to deal with stabilization and robustness. These intensive developments are a testimony of the vitality and the maturity of this field that remains an area of active research; see for instance recent works among others (Bara, 2011; Ferranti, De Zutter, Knockaert, & Dhaene, 2011; Shi & Yan, 2011; Steinbrecher, Stykel, Hinze, & Kunkel, 2012).

In this paper, our focus is on discrete singular systems under positivity constraint on their states. This is inherent to many real-world systems for which the states are intrinsically nonnegative since they can represent real physical quantities such as concentrations, level and volume of matter transfer, size of populations, etc. Singular systems which have nonnegative states whenever the initial conditions are nonnegative are referred to as *positive systems* (Farina & Rinaldi, 2000; Kaczorek, 2005; Luenberger, 1979) or evenly as *nonnegative systems* (Chellaboina, Haddad, & Hui, 2010).

Although many fundamental issues have been well-investigated for standard singular systems and, in particular, for standard positive systems, they have not been sufficiently investigated for the specific class of positive singular systems. To the best of our knowledge few works on such systems can be found in the literature (Bru, Coll, Romero-Vivo, & Sánchez, 2003; Bru, Coll, & Sánchez, 2002; Cantó, Coll, & Sánchez, 2008; Herrero, Ramírez, & Thome, 2007, 2010; Reis & Virnik, 2009; Virnik, 2008b). These works are based on a common standing, but unnecessary, assumption for positivity of a singular system. That is, the matrix that represents the projector on the set of admissible initial conditions is nonnegative. Most of the reported results have focused on other fundamental properties such as reachability and controllability. The stability issue was considered only in Ait Rami and Napp (2012) and Virnik (2008b).

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In this paper, unlike the previous reported results on discrete positive singular systems, positivity is fully investigated without any unnecessary assumption. Note that, in general, numerically checking positive invariance for a particular set of initial conditions even for a standard linear system can be quite complicated. This fact also applies to the stability analysis for a given set of initial conditions. This issue has been investigated in Nieuwenhuis (1984) and Stern (1982) for LTI systems with respect to a closed convex pointed cone for which necessary and sufficient conditions for positive invariance and stability have been provided. However, the proposed results are rather theoretical and cannot be checked numerically. In the case of positive singular systems we have to deal with a specific conic set of the form $\mathrm{im}(P)\cap\mathbb{R}^n_+$, where the matrix P represents the projector on the admissible set of initial conditions. For such a set the stability issue is also addressed. The proposed approach is numerically appealing for checking positivity and stability of a given discrete singular system. All the proposed conditions are necessary and sufficient and can be checked by using linear programming (LP).

The structure of the paper is as follows. Section 2 gives the necessary background on singular systems. Section 3 is concerned with the positivity of discrete singular systems for which some characterizations are provided together with an illustrative example. Section 4 deals with the stability issue. In Section 5 the notion of internal positivity is investigated. Section 6 gives some conclusions.

Notation: $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$ and $\dot{\mathbb{R}}^n_+$ its interior. A real matrix (or a vector) $M = [M(i,j)]$ is called nonnegative, denoted by $M \geq 0$, if all its components are nonnegative (i.e., $M(i,j) \geq 0$); analogously, a positive matrix or vector is denoted by $M > 0$ if its components are strictly positive. M^+ is used to denote the Moore–Penrose pseudoinverse of the matrix M and $\sigma(M)$ its spectrum.

2. Solvability

This section provides preliminary results regarding the existence and characterization of the solution of the following time-invariant homogeneous singular system:

$$Ex(k+1) = Ax(k) \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$. In contrast to standard linear systems for which E is invertible, system (1) may not possess a solution for arbitrary initial conditions.

Definition 2.1. The set of initial conditions for which system (1) has a solution is called the *set of admissible initial conditions*.

The characterization of the admissible set of initial conditions with their associated trajectories involves the Drazin inverse. Hence, we first present some basic properties of this kind of inverse (see Campbell & Meyer, 1991, Drazin, 1958 for more details). For any matrix $M \in \mathbb{R}^{n\times n}$ there always exists a unique matrix M^D, which is called the Drazin inverse of M, such that $M^DM = MM^D$, $M^DMM^D = M^D$ and $M^DM^{\nu+1} = M^\nu$, where ν is the smallest nonnegative integer such that $\mathrm{rank}(M^\nu) = \mathrm{rank}(M^{\nu+1})$. In Kunkel and Mehrmann (2006), it is shown how the Drazin inverse can be computed; see also Cantó, Coll, and Sánchez (2005), Yimin (1996) and Zhang (2001) for more details on this issue. One way to compute it is as follows: by using the Jordan canonical form, any matrix M can be decomposed as

$$M = T\begin{pmatrix} C & 0 \\ 0 & N\end{pmatrix}T^{-1}, \tag{2}$$

where C is invertible and N is a nilpotent matrix. Then, its Drazin inverse is given by

$$M^D = T\begin{pmatrix} C^{-1} & 0 \\ 0 & 0\end{pmatrix}T^{-1}. \tag{3}$$

The following result presents a characterization for the solvability of system (1). In Campbell (1980), a precise explicit solution to system (1) has been given (see also Kunkel & Mehrmann, 2006).

Theorem 2.2 (Campbell, 1980)**.** *The singular system (1) admits a unique solution for each admissible initial condition if and only if (E, A) is regular (i.e., there exists a $\lambda \in \mathbb{C}$ such that $(\lambda E - A)^{-1}$ exists). Moreover, the set of admissible initial conditions is given by $\mathcal{X}_0 := \mathrm{im}(\widehat{E}^D\widehat{E})$ and the solutions of (1) have the following form:*

$$x(k) = (\widehat{E}^D\widehat{A})^k\widehat{E}^D\widehat{E}v, \tag{4}$$

where v is an arbitrary vector in $\mathbb{R}^n$, the matrices $\widehat{A}$ and $\widehat{E}$ are given by

$$\widehat{E} = (\lambda E - A)^{-1}E, \qquad \widehat{A} = (\lambda E - A)^{-1}A, \tag{5}$$

with λ any complex number such that $(\lambda E - A)^{-1}$ exists, and $\widehat{E}^D$ is the Drazin inverse of $\widehat{E}$.

Remark 2.3. Theorem 2.2 summarizes the results of Theorems 3.6.1 and 3.6.2 in Campbell (1980). Based on this theorem, one can see that the trajectory (4) is the solution to the difference equation $x(k+1) = \widehat{E}^D\widehat{A}x(k)$ with $x(0) = \widehat{E}^D\widehat{E}v \in \mathrm{im}(\widehat{E}^D\widehat{E})$. Note that the solution (4) does not depend on the value of λ used to define $\widehat{E}$ and $\widehat{A}$. For more details see Campbell (1980) and Kunkel and Mehrmann (2006).

According to Theorem 2.2 we assume throughout the rest of the paper that (E, A) is regular.

In the sequel, we shall make use of some useful properties of the matrices $P := \widehat{E}^D\widehat{E}$ and $\bar{A} = \widehat{E}^D\widehat{A}$ that characterize the admissible set of initial conditions. Such properties are presented in the following result.

Lemma 2.4 (Ait Rami & Napp, 2012, Lemma 3.2)**.** *The following properties hold true.*

(i) *P is idempotent or a projector (i.e., $P^2 = P$).*
(ii) *$P\bar{A} = \bar{A}P = \bar{A}$.*
(iii) *For any solution $x(k)$ to system (1) we have*

$$Px(k) = x(k).$$

3. Positivity

This section deals with the characterization of positivity of system (1). Although the positivity analysis is simple when the set of admissible initial conditions is the positive orthant, the characterization of positivity for an arbitrary set of initial conditions is not, in general, an easy task. The positive invariance for LTI systems with respect to a given cone has been studied in Nieuwenhuis (1984) and Stern (1982). However, the reported results are rather theoretical and cannot be checked numerically. In what follows, we shall investigate computationally sound conditions for the positivity of system (1) in connection with the conic set of the form $\mathrm{im}(P)\cap\mathbb{R}^n_+$. Observe that when the matrix E is nonsingular, system (1) reduces to the standard linear system $x(k+1) = E^{-1}Ax(k)$. Obviously, in this case system (1) is positive if and only if $E^{-1}A$ is a nonnegative matrix (i.e., $E^{-1}A \geq 0$).

Definition 3.1. We say that system (1) is *positive* if for any nonnegative admissible initial condition $x(0) \in \mathcal{X}_0 = \mathrm{im}(P)\cap\mathbb{R}^n_+$ we have that $x(k) \geq 0$ for all $k \geq 0$.

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