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Brief paper

Solution for state constrained optimal control problems applied to power split control for hybrid vehicles☆

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ABSTRACT

This paper presents a numerical solution for scalar state constrained optimal control problems. The algorithm rewrites the constrained optimal control problem as a sequence of unconstrained optimal control problems which can be solved recursively as a two point boundary value problem. The solution is obtained without quantization of the state and control space. The approach is applied to the power split control for hybrid vehicles for a predefined power and velocity trajectory and is compared with a Dynamic Programming solution. The computational time is at least one order of magnitude less than that for the Dynamic Programming algorithm for a superior accuracy.

1. Introduction

This paper concerns numerical solutions for convex scalar state constrained optimal control problems. In the literature, several approaches are known to solve problems of this type.

Firstly, the problem can be solved "directly" with, e.g., multiple shooting (Sager, 2005), or other approximate methods that require a quantization, see Gerdts (2008), Gerdts and Kunkel (2008), Wang, Gui, Teo, Loxton, and Yang (2009) and Yu, Li, Loxton, and Teo (2012). Dynamic Programming (DP) (Bertsekas, 2000) is another approach often applied to optimal control problems of low dimension. Drawback of the quantization of control and state variables is that it only leads to an approximation of the original problem. This results in the classical trade-off between accuracy and computational demand induced by the quantization grid size.

Secondly, indirect methods derive the optimal solution using a two step procedure; as a first step the Pontryagin Minimum Principle (PMP) (Pontryagin, Boltyanskii, Gamkrelidze, & Mischenko, 1962) is applied to derive the necessary conditions for optimality in the form of a differential equation on the costate variable and the static optimization of the Hamiltonian function. These analytical results allow one to write the unconstrained optimal control problem into a boundary value problem. As a second step, the remaining boundary value problem can then be solved, without quantization of the state and control space, but generally using the discretization of the time space, using information about the state and costate at the boundaries. Extensions to the state constrained optimal control case can be found (Fabien, 1996; Jacobson & Lele, 1967). These solutions involve, e.g., a penalty function which comes with the cost of an increased state dimension, however.

This contribution presents a novel numerical approach for convex scalar optimal control problems with "pure" state constraints which has superior results in terms of computational demand and accuracy compared to other known numerical techniques for problems of this type. It takes advantage of the PMP, it does not require quantization of the state and control space, it includes state constraints, state dependent losses and non-smooth cost function descriptions, all without the introduction of a penalty function. A proof for optimality of the solution is included. The novel approach is applied to the power split control problem in hybrid vehicles and is bench marked, for computation time and accuracy, with a DP algorithm. The influence of battery voltage increase as a function of the battery state-of-energy is also evaluated.

The paper is organized as follows. In Section 2, the novel numerical algorithm based on the PMP is given. Section 3 introduces the power split control problem for hybrid vehicles. A comparison of the novel algorithm with the DP algorithm is presented in Section 4. Conclusions can be found in Section 5.

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2. Problem description

Consider an optimal control problem with a scalar state and control:

$$(\mathcal{P}_0)\begin{cases}\min\limits_{u\in\mathcal{U}} \int_{t_0}^{t_1} F(t,x(t),u(t))dt \\ \text{subject to:}\quad \dot{x} = f(t,x(t),u(t)), \\ x(t_0)=x_0, \qquad x(t_1)=x_1, \\ h(t,x(t)) = \begin{bmatrix} x(t)-\bar{x}(t) \\ \underline{x}(t)-x(t)\end{bmatrix} \leq \begin{bmatrix}0\\0\end{bmatrix},\end{cases}$$

here, u is the control variable, $\mathcal{U}$ is a closed convex set of admissible controls for every t and x, t_0 the initial time, t_1 the end time, F the time dependent cost function, x the state variable, f the state dynamics equation, x_0 and x_1 the boundary conditions, h the inequality constraint on the state, $\bar{x}$ the upper state constraint, and $\underline{x}$ the lower state constraint. It is assumed that the functions F, f and h are continuous in all their arguments, and continuously differentiable in x.

2.1. Necessary conditions of optimality

In this section, the necessary conditions for optimality for problem $\mathcal{P}_0$ are given (Hartl, Sethi, & Vickson, 1995; Maurer, 1977; Seierstad & Sydsæter, 1987). The Hamiltonian is defined with:

$$H(t,x(t),u(t),p(t)) = F(t,x(t),u(t)) - p(t)f(t,x(t),u(t)). \quad (1)$$

The state inequality constraints can be adjoined to the Hamiltonian to form the following Lagrangian:

$$L(t,x(t),u(t),p(t),\lambda(t)) = H(t,x(t),u(t),p(t)) + \lambda^{\top}(t)h(t,x(t)). \quad (2)$$

Applying the PMP as in Theorem 9.3.1 of Vinter (2000, p. 339), it follows that if the control is optimal, then there exists a nontrivial piecewise continuous multiplier function $p(t) \not\equiv 0$ such that the following conditions are satisfied:

- the differential equation on the adjoint multiplier function:

$$\dot{p}(t) = \frac{\partial L}{\partial x} \quad (3)$$

- the complementary slackness condition:

$$\lambda_1(t) = 0 \quad \text{for } t \in \left[v : x^*(v) < \bar{x}\right], \quad (4)$$

$$\lambda_2(t) = 0 \quad \text{for } t \in \left[v : x^*(v) > \underline{x}\right], \quad (5)$$

- the condition on the adjoint multiplier, see also Hartl et al. (1995, Theorem 4, p. 186), for $t_a < t_b$ in $[t_0, t_1]$:

$$p(t_b^+) - p(t_a^+) = \int_{t_a}^{t_b} \dot{p}(t)dt + \int_{(t_a,t_b]} \frac{\partial h}{\partial x} d\xi_1(t) - \int_{(t_a,t_b]} \frac{\partial h}{\partial x} d\xi_2(t), \quad (6)$$

where ξ_1 and ξ_2 are of bounded variation, non-increasing, constant on intervals where $\underline{x} < x < \bar{x}$, right continuous and have left-sided limits everywhere. The multiplier trajectory p has a discontinuity given by the following jump condition:

$$p(\tau^+) = p(\tau^-) + \mu_1(\tau) - \mu_2(\tau), \quad (7)$$

with $\mu_1 \geq 0$ and $\mu_2 \geq 0$. Under the assumption that ξ_1 and ξ_2 have a piecewise continuous derivative, it is possible to set $\lambda_1(t) = \dot{\xi}_1(t)$, $\lambda_2(t) = \dot{\xi}_2(t)$, for every t for which ξ_1 and ξ_2 exist and $\mu_1(\tau) = \xi_1(\tau^-) - \xi_1(\tau^+)$, $\mu_2(\tau) = \xi_2(\tau^-) - \xi_2(\tau^+)$, for all $\tau \in [t_0, t_1]$ where ξ_1 and ξ_2 are not differentiable,

- the Hamiltonian H has a global minimum with respect to control u:

$$u^*(t) = \arg\min_{u\in\mathcal{U}} H(t,x^*(t),u(t),p^*(t)), \quad (8)$$

where $x^*(t)$ is the optimal state trajectory, $u^*(t)$ the optimal control trajectory, $p^*(t)$ the corresponding adjoint multiplier function.

2.2. Numerical solution for the unconstrained problem

In this section, a numerical solution for the unconstrained problem is discussed. Here "unconstrained" refers to a problem of type $\mathcal{P}_0$ without the inequality constraint. To derive a well defined two point boundary value problem from conditions (3) and (8), the following property is required.

Lemma 1. *Let $H = F - pf$ with $p \geq 0$, x a scalar variable, F a convex function in u, and f a strictly concave and strictly monotonic decreasing function in u, additionally, assume the differential equation $\dot{p} = \frac{\partial H}{\partial x}$, with $\frac{\partial H}{\partial x}$ locally Lipschitz in p on a domain defined by $\mathcal{U}$, then the solution u^* of (8) is a monotonic decreasing function of p and there is a monotonic increasing relation between the initial value of the multiplier $p(t_0)$ and the final state $x(t_1)$ and between the inverse relation of the final state $x(t_1)$ and the initial value of the multiplier $p(t_0)$.*

Proof. The proof uses the convexity properties and the existence and uniqueness of the solution of the differential equation (3). If f is strictly concave in u, and $p > 0$, then the function $-pf$ is strictly convex in u. The sum of two convex functions is also convex. So, if F is convex and $-pf$ strictly convex, then H is strictly convex in u. If H is strictly convex in u, then it has a unique minimum defined by $\frac{\partial F}{\partial u} - p\frac{\partial f}{\partial u} = 0$.

If f is strictly concave and strictly monotonic decreasing, then $\frac{\partial f}{\partial u} < 0$ and also strictly monotonic decreasing. In the optimum it holds that $p = \frac{\partial F}{\partial u} / \frac{\partial f}{\partial u}$ with $\frac{\partial f}{\partial u} < 0$ and $p \geq 0$, then $\frac{\partial F}{\partial u} \leq 0$, and, because F is convex, $\frac{\partial F}{\partial u}$ is also monotonic increasing. Again using $p = \frac{\partial F}{\partial u} / \frac{\partial f}{\partial u}$ with $\frac{\partial F}{\partial u} \leq 0$ and monotonic increasing and $\frac{\partial f}{\partial u} < 0$ and strictly monotonic decreasing, it follows that the minimum u^* of H is a monotonic decreasing function of p.

If $\frac{\partial H}{\partial x}$ is Lipschitz continuous in p, it follows that p is a unique solution of (3), see Khalil (2002, Theorem 3.1), hence $p_a(t_0) > p_b(t_0)$ implies $p_a^0(t) > p_b^0(t)$, where $p^0(t)$ denotes the trajectory resulting from $p(t_0)$. Using that the minimum u^* of H is a monotonic decreasing function of p, it follows that $u_a^0(t) \leq u_b^0(t)$ if $p_a^0(t) > p_b^0(t)$, where u^0 denotes the control trajectory resulting from $p(t_0)$. Finally, given the state dynamics $\dot{x} = f$ with f monotonic decreasing in u, a monotonic increasing relation is found between $p(t_0)$ and $x(t_1)$ and likewise between $x(t_1)$ and $p(t_0)$. □

Given the necessary conditions of optimality, the following boundary value problem is obtained:

$$(\mathcal{P}_{BVP})\begin{cases}\dot{x}(t) = f(t,x(t),u^*(t)), \\ \dot{p}(t) = \dfrac{\partial H(t,x(t),u^*(t),p(t))}{\partial x}, \\ x(t_0) = x_0, \qquad x(t_1) = x_1,\end{cases}$$

in which $u^*(t)$ is the solution of (8). Using single shooting, an initial value problem associated with this boundary value problem can be derived. Generally, we have to sample the time space and apply numerical integration methods to solve this initial value problem, e.g., the Euler scheme, where the discrete time sample moments are indicated by variable $k = [1, \ldots, n]$ with length $n \in \mathbb{N}$ defined by $t_1 - t_0$ with equidistant step size Δt:

$$\Delta t = \frac{t_1 - t_0}{n - 1}. \quad (9)$$