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Brief paper

# An extension of the prediction scheme to the case of systems with both input and state delay☆

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ABSTRACT

In this contribution we present an extension of the prediction scheme proposed in Manitius and Olbrot (1979) for the compensation of the input delay to the case of linear systems with both input delay and state delay. For simplicity of the presentation we treat the case of systems with one state delay.



## 1. Introduction

In the Introduction Chapter of the recently published book (Krstic, 2009) it is stated that "the area of control design for systems with simultaneous input and state delay is underdeveloped". At the same time, it is mentioned there that the stabilizing problems for systems with state delay only "are the easiest in our list as they can be solved using finite dimensional feedback laws". In this contribution we present an extension of the prediction scheme proposed in Manitius and Olbrot (1979) for the compensation of the input delay in the computation of stabilizing controllers for linear systems with both input delay and state delay. For simplicity of the presentation we treat the case of systems with one state delay, but the presented results can be extended to the case of systems with multiple state delays, as well.

In Section 2 we provide basic notations used in the contribution and give the formulation of the stabilization problem for systems with input and state delay. Section 3 is devoted to the computation of the stabilizing control laws. Similar to the case of systems with input delay we start with an explicit expression for the solution of an initial value problem for a time-delay system. Then, we apply this expression for the computation of future states in the form of functionals that depend on the present and past states of the time-delay system. And, finally, we compute the desired stabilizing control law. The stabilizing law is of the form of an integral equation, similar to that obtained in Manitius and Olbrot (1979), with some additional terms due to the presence of the state delay in the system. Section 4 is dedicated to the stability analysis of the closed-loop system. The principal result of the section is an upper exponential estimate for the solutions of the closed-loop system. In Section 5 some basic results concerning the complete type functionals for an exponentially stable system are given. In Section 6 we present a Lyapunov–Krasovskii type stability analysis of the closed-loop system. The key element of the analysis is a simple modification of the backstepping transformation of the control variable proposed in Krstic and Smyshlyaev (2008). This transformation allows us to present the closed-loop system in a form more appropriate for the consequent stability analysis. Here we propose for the transformed system a Lyapunov functional, similar to that of Krstic (2009), with a single modification: we use a complete type functional instead of the quadratic Lyapunov form used in Krstic (2009). As a result we obtain an upper exponential estimate for the solutions of the transformed system and derive a similar exponential estimate for the original control variable. Several examples illustrating the computation of the stabilizing control laws are given in Section 7.

## 2. Problem formulation

Given a time-delay system of the form

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t-h) + Bu(t-\tau), \qquad (1)$$

where $A_j$, $j = 0, 1$, are real $n \times n$ matrices, and $B$ is a real $n \times m$ matrix. The system delays satisfy the inequalities $0 < h \le \tau$. The opposite case, $\tau < h$, can be treated similarly with trivial modifications. Let $t_0 \ge 0$ be an initial time instant and $\varphi : [-h, 0] \to R^n$

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be an initial function. We assume that the function belongs to the space of piece-wise continuous functions, $PC([-h,0],R^n)$, defined on the segment $[-h,0]$. Let $x(t,t_0,\varphi)$ stand for the solution of system (1) under the initial condition

$$x(t_0+\theta,t_0,\varphi)=\varphi(\theta),\quad \theta\in[-h,0],$$

and $x_t(t_0,\varphi)$ denote the restriction of the solution to the segment $[t-h,t]$

$$x_t(t_0,\varphi):\theta\to x(t+\theta,t_0,\varphi),\quad \theta\in[-h,0].$$

We omit arguments $t_0$ and $\varphi$ in these notations and write $x(t)$ and $x_t$ instead of $x(t,t_0,\varphi)$ and $x_t(t_0,\varphi)$, when no confusion may arise.

The euclidean norm is used for vectors, and the induced matrix norm for matrices. For elements of the space $PC([-h,0],R^n)$ we use the uniform norm

$$\|\varphi\|_h=\sup_{\theta\in[-h,0]}\|\varphi(\theta)\|.$$

In the following we assume that there exist matrices $F_0$ and $F_1$, such that the system

$$\frac{dx(t)}{dt}=(A_0+BF_0)\,x(t)+(A_1+BF_1)\,x(t-h) \tag{2}$$

is exponentially stable.

**Problem**: Find a control law under which system (1), for $t\geq\tau$, coincides with (2).

## 3. General scheme

Let us denote by $K(t)$ the fundamental matrix of system (1); see Bellman & Cooke, 1963. The matrix satisfies the equation

$$\frac{d}{dt}K(t)=A_0K(t)+A_1K(t-h),\quad t\geq 0,$$

and the initial conditions: $K(t)=0_{n\times n}$, $t<0$, $K(0)=I$.

Given an initial time instant $t_0\geq 0$, and an initial function $\varphi\in PC([-h,0],R^n)$, then the corresponding solution of system (1) can be written as

$$x(t,t_0,\varphi)=K(t-t_0)\varphi(0)+\int_{-h}^{0}K(t-t_0-\theta-h)A_1\varphi(\theta)d\theta + \int_{t_0}^{t}K(t-\xi)Bu(\xi-\tau)d\xi,\quad t\geq t_0.$$

In particular this means that

$$x(t+\tau)=K(\tau)x(t)+\int_{-h}^{0}K(\tau-\theta-h)A_1x(t+\theta)d\theta + \int_{-\tau}^{0}K(-\xi)Bu(t+\xi)d\xi,\quad t\geq 0, \tag{3}$$

and

$$x(t+\tau-h)=K(\tau-h)x(t)+\int_{-h}^{0}K(\tau-\theta-2h)A_1x(t+\theta)d\theta + \int_{-\tau}^{-h}K(-h-\xi)Bu(t+\xi)d\xi,\quad t\geq 0. \tag{4}$$

We start with a control law of the form

$$u(t)=F_0x(t+\tau)+F_1x(t+\tau-h),\quad t\geq 0,$$

where $x(t+\tau)$ and $x(t+\tau-h)$ in the right-hand side of the preceding expression are replaced by (3) and (4), respectively. As a result we arrive at a control law of the form

$$u(t)=f(u_t,x_t),\quad t\geq 0,$$

where

$$u_t:\xi\to u(t+\xi),\quad \xi\in[-\tau,0],$$

and the functional $f(\psi,\varphi)$ is defined on the direct product of the functional spaces $PC([-\tau,0],R^m)\times PC([-h,0],R^n)$ as follows:

$$\begin{aligned} f(\psi,\varphi) = {} & [F_0K(\tau)+F_1K(\tau-h)]\,\varphi(0) \\ & + \int_{-h}^{0}F_0K(\tau-\theta-h)A_1\varphi(\theta)d\theta \\ & + \int_{-h}^{0}F_1K(\tau-\theta-2h)A_1\varphi(\theta)d\theta \\ & + \int_{-\tau}^{0}F_0K(-\xi)B\psi(\xi)d\xi \\ & + \int_{-\tau}^{-h}F_1K(-h-\xi)B\psi(\xi)d\xi. \end{aligned}$$

In the explicit form this control law is given by the integral equation

$$\begin{aligned} u(t) = {} & \int_{-\tau}^{0}F_0K(-\xi)Bu(t+\xi)d\xi \\ & + \int_{-\tau}^{-h}F_1K(-h-\xi)Bu(t+\xi)d\xi \\ & + \int_{-h}^{0}F_0K(\tau-\theta-h)A_1x(t+\theta)d\theta \\ & + \int_{-h}^{0}F_1K(\tau-\theta-2h)A_1x(t+\theta)d\theta \\ & + [F_0K(\tau)+F_1K(\tau-h)]\,x(t),\quad t\geq 0. \end{aligned} \tag{5}$$

**Remark 1.** For $h=0$ system (1) is of the form

$$\frac{dx(t)}{dt}=(A_0+A_1)\,x(t)+Bu(t-\tau),$$

and $K(t)=e^{(A_0+A_1)t}$. In this case Eq. (5)

$$u(t)=Fe^{(A_0+A_1)\tau}+\int_{t-\tau}^{t}Fe^{(A_0+A_1)(t-\theta)}Bu(\theta)d\theta,$$

where $F=F_0+F_1$ coincides with the classical prediction control obtained through various different approaches (Artstein, 1982; Kwon & Pearson, 1980; Manitius & Olbrot, 1979).

A particular solution of the integral equation is defined by an initial function $\psi\in PC([-\tau,0],R^m)$,

$$u(t)=\psi(t),\quad t\in[-\tau,0).$$

The control law (5) is such that for $t\in[0,\tau)$ system (1) under this control is of the form

$$\frac{dx(t)}{dt}=A_0x(t)+A_1x(t-h)+B\psi(t-\tau),$$

while for $t\geq\tau$ the system coincides with system (2).

**Remark 2.** It may happen that the spectrum of the closed-loop system is finite. One such case has been reported in Bekiaris-Liberis and Krstic (2010). Here we observe that if a matrix $A_1$ is such that $A_1+BF_1=0$, then the spectrum of the closed-loop system (1), (5) coincides with that of the matrix $A_0+BF_0$.

## 4. Exponential estimates

Any particular solution of the closed-loop system (1), (5) is defined by the corresponding initial conditions

$$x(t)=\varphi(t),\quad t\in[-h,0],\ \varphi\in PC([-h,0],R^n),$$
$$u(t)=\psi(t),\quad t\in[-\tau,0),\ \psi\in PC([-\tau,0],R^m).$$