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Brief paper

Stability certification of large scale stochastic systems using dissipativity☆

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ABSTRACT

In this paper, we analyse the stability of large-scale nonlinear stochastic systems, represented as an interconnection of lower-order stochastic subsystems. Stochastic stability in probability and noise-to-state stability are addressed, and sufficient conditions for the latter are provided. The method proposed proves network stability by using appropriate stochastic passivity properties of its subsystems, and the structure of its interactions. Stability properties are established by the diagonal stability of a dissipativity matrix, which incorporates information about the passivity properties of the systems and their interconnection. Next, we derive equilibrium-independent conditions for the verification of the relevant passivity properties of the subsystems. Finally, we illustrate the proposed approach on a class of biological reaction networks.

1. Introduction

The analysis of nonlinear systems becomes intractable as the dimension of the state space increases. It is imperative to develop approaches that decompose this analysis into smaller subproblems. In this paper, we represent large-scale nonlinear stochastic systems as an interconnection of lower-order stochastic dynamical subsystems. We then certify stability based on appropriate stochastic passivity properties of the subsystems and the structure of their interactions.

Previous studies have shown the effectiveness of this approach for deterministic models of biological networks, Arcak and Sontag (2006, 2008). In Arcak and Sontag (2006), global asymptotic stability of a cyclic interconnection structure is established from the diagonal stability of a dissipativity matrix that incorporates information about the passivity properties of the subsystems and the interconnection structure of the network. The results are extended in Arcak and Sontag (2008) to a more general interconnection structure. Both Arcak and Sontag (2006, 2008) exploit output strict passivity (OSP) properties and corresponding storage functions of the subsystems, and construct a composite Lyapunov function for the interconnection using these storage functions.

In biochemical reactions, deterministic models may be inadequate, particularly when the copy numbers of the species are small. Stochasticity appears as external noise (due to cell-to-cell variability of external signals) and as intrinsic noise (since chemical reactions depend on random motion). While external noise can be incorporated in noise-driven deterministic models, *i.e.* stochastic differential equations (SDEs), internal noise is accounted for by a Chemical Master Equation models (CME). Under appropriate assumptions, Gillespie (2000), it is common to perform a diffusion approximation of the CME, leading to the Chemical Langevin Equation (CLE), which is a particular type of SDE. Thus, both internal and external noise can be treated jointly with SDEs.

We study large-scale nonlinear stochastic models described by SDEs. We extend the passivity approach in Arcak and Sontag (2006, 2008) to the stochastic framework, by using the expansion of the definitions of passivity introduced in Florchinger (1999). We prove stability in probability for an interconnection of stochastic OSP (sOSP) subsystems, if an appropriate diagonal stability condition holds for a dissipativity matrix similar to the one in Arcak and Sontag (2006). Early references, such as Michel (1975), Michel and Rasmussen (1976), constructed composite Lyapunov functions for stochastic stability. However, as is common in the classical large-scale systems literature, these references restrict the magnitude of the coupling terms without regard to their sign structure. The

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passivity-based approach in the present paper takes advantage of the negative feedback loops in the network to obtain less restrictive stability criteria.

We next investigate the notion of Noise-to-State stability (NSS), defined in Krstic and Deng (1998), which is a stochastic counterpart of deterministic input-to-state stability, Sontag (1989). NSS implies that if there exists a bound in the noise variance, the state of a NSS system is also bounded in probability. This notion is less restrictive than stochastic stability in the sense that it accommodates systems with nonvanishing noise at the equilibrium, and unknown noise intensity. First, we provide a new sufficient condition for NSS that is easy to verify. We then introduce a new input–output definition that combines NSS and OSP properties, referred to as NSS⊕OSP. We show that the interconnected system is NSS if the diagonal stability of a similar dissipativity matrix is ascertained.

Since passivity properties are defined in reference to the equilibrium, which depends on the full network, the verification of the sOSP and NSS⊕OSP properties of the subsystems is a difficulty encountered in the methodology presented. In Arcak and Sontag (2006), equilibrium-independent results for the verification of OSP properties are provided. In this paper, we provide an extension for stochastic systems. We derive equilibrium-independent verification of stochastic passivity properties, which are not considered in the classical literature.

In Section 2, we provide the necessary notation and definitions, and derive sufficient conditions for NSS. The main results for stochastic stability of interconnected systems are presented in Section 3, where stability in probability and noise-to-state stability are achieved. In Section 4, we focus on the input/output passivity properties of the systems, by deriving equilibrium-independent conditions that guarantee sOSP and NSS⊕OSP. Finally, in Section 5, we illustrate the application of the results obtained to classes of biological reaction networks.

2. Preliminaries

Consider the following nonlinear stochastic system

$$dx = f(x)dt + l(x)\Sigma dw \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t)$ is an r-dimensional independent standard Wiener process, $\Sigma = \{\sigma_{ij}\}$ is an $r \times r$ non-negative-definite matrix, and σ_{ij} represents the intensity with which the jth source of uncertainty influences the ith state. Assume that the vector field and matrix function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $l : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz continuous.

For the notions of stochastic stability and passivity, defined in Sections 2.1 and 2.2, we assume $\Sigma = I$ because $l(x)$ can be redefined to incorporate the constant Σ. Moreover, we also assume $f(0) = 0$ and $l(0) = 0$, so that $x(t) \equiv 0$ is a solution for the system. However, for the notion of noise-to-state stability, defined in Section 2.3, where Σ is treated as an unknown, the assumption $\Sigma = I$ is dropped, and also $l(x)$ is not necessarily required to be vanishing at the origin ($l(0) \neq 0$).

Notation and definitions. For a matrix $A \in \mathbb{R}^{p\times q}$, the *Frobenius Norm*, $|\cdot|_F : \mathbb{R}^{p\times q} \to \mathbb{R}_{\geq 0}$, is defined as $|A|_F = \sqrt{\mathrm{Tr}\{A^TA\}} = \sqrt{\sum_{i=1}^{p}\sum_{j=1}^{q}|a_{ij}|^2}$. A scalar continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is *class* $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty}\alpha(s) = \infty$. A scalar continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *class* $\mathcal{KL}$ if, for each fixed t, function $\beta(\cdot, t)$ is class $\mathcal{K}$ and, for each fixed s, function $\beta(s, \cdot)$ is decreasing and $\lim_{t\to\infty}\beta(s, t) = 0$. Given functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$, the expression $a(s) = O(b(s))$ as $s \to \infty$ means that $\exists M > 0 \in \mathbb{R}$ and $\exists x_0 \in \mathbb{R}$ such that $|a(x)| \leq M|b(x)|\ \forall x > x_0$. Analogously, the expression $a(s) = o(b(s))$ as $s \to \infty$ means that $\forall M > 0 \in \mathbb{R}$, $\exists x_0 \in \mathbb{R}$ such that $|a(x)| \leq M|b(x)|\ \forall x > x_0$, or equivalently, $\lim_{s\to\infty}|a(s)/b(s)| = 0$. Given a continuous function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, we denote by:

$$\bar{f}(x) = \sup_{|s|\leq|x|} f(s) \quad \text{and} \quad \underline{f}(x) = \inf_{|s|\geq|x|} f(s). \tag{2}$$

Clearly, $\bar{f}$ and $\underline{f}$ are nondecreasing functions. Note that, $f^2(x) : x \to f(x)f(x)$, and so $\overline{f^2}(x) : x \to \overline{f(x)f(x)}$.

2.1. Stochastic stability

An extensive coverage of stochastic stability and stochastic Lyapunov theorems exists in the literature, Hasminskii (1980); Kushner (1967). In what follows, we refer to Deng, Krstic, and Williams (2001) where a notation based on class $\mathcal{K}$ functions is used, instead of the classical $\epsilon - \delta$.

Definition 2.1. The equilibrium $x = 0$ of system (1) is:

(i) Globally Stable in Probability if $\forall\epsilon > 0, \exists\gamma \in \mathcal{K}$ s.t.

$$P\{|x(t)| \leq \gamma(|x_0|)\} \geq 1 - \epsilon, \quad \forall t \geq 0, \forall x_0 \in \mathbb{R}^n. \tag{3}$$

(ii) Globally Asymptotically Stable in Probability if it is globally stable in probability and

$$P\left\{\lim_{t\to\infty}|x(t)| = 0\right\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{4}$$

Proposition 2.2. *For system (1), with $\Sigma = I$, suppose there exists a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions α_1, α_2, and a continuous nonnegative function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that for all $x \in \mathbb{R}^n$,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5}$$

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x}f(x) + \frac{1}{2}Tr\left\{l(x)^T\frac{\partial^2 V}{\partial x^2}l(x)\right\} \leq -S(x). \tag{6}$$

Then, the equilibrium $x = 0$ is globally stable in probability. If S is a positive definite function, the equilibrium $x = 0$ is globally asymptotically stable in probability.

2.2. Stochastic passivity and output strict passivity

Consider now the controlled stochastic nonlinear system

$$\begin{cases} dx = (f(x) + g(x)u)dt + l(x)\Sigma dw \\ y = h(x) \end{cases} \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^m$ is the output.[2]

Definition 2.3. The system (7), with $\Sigma = I$, is said to be stochastic passive, Florchinger (1999), if there exists a $\mathcal{C}^2$ positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that $\forall x \in \mathbb{R}^n$, and $\forall u \in \mathbb{R}^m$,

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x}(f(x) + g(x)u) + \frac{1}{2}\mathrm{Tr}\left\{l(x)^T\frac{\partial^2 V}{\partial x^2}l(x)\right\} \leq h(x)^Tu - S(x) \tag{8}$$

[2] The control input u may be seen as a function of t that satisfies appropriate regularity poperties so as to obtain existence and uniqueness of solutions. However, we do not need to specify these regularity properties in this paper, since the only place where inputs are used is in defining passivity and other stability properties. These properties are defined in terms of algebraic inequalities involving Lyapunov-like functions and only pointwise values of x and u, so that regularity of $u(t)$ as a function of t is not relevant. On the other hand, when interconnecting several systems, $u(t)$ becomes a function of the subsystems' state variables, and the closed-loop system is assumed to satisfy the conditions assumed for (1).