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Technical communique

# Cyclic invariance for discrete time-delay systems☆

Warody Lombardi[a,b], Sorin Olaru[a], Georges Bitsoris[c], Silviu-Iulian Niculescu[b]

[a] SUPELEC Systems Sciences (E3S) - Automatic Control Department, Gif-sur-Yvette, France
[b] LSS - Laboratory of Signals and Systems, SUPELEC-CNRS, Gif-sur-Yvette, France
[c] Department of Electrical and Computer Engineering - University of Patras, Patras, Greece

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ABSTRACT

This technical communique introduces a new concept of set invariance with respect to linear discrete time dynamics affected by delay. We are interested in the definition and characterization of sequences of cyclically invariant subsets in the state space. The algebraic conditions established in the late '80s for linear dynamics are generalized to invariance analysis in the presence of delays for given sequences of polyhedral sets.



## 1. Introduction

The present paper concentrates on the set invariance of polyhedral sets with respect to discrete time dynamical systems affected by delay. It is known that set invariance in the original state space can lead to conservative definitions through its delay-independent characteristic (Hennet & Tarbouriech, 1998; Vassilaki & Bitsoris, 1999), while the augmented state space can lead to less conservative but complex constructions (Gielen, Lazar, & Kolmanovsky, 2010; Lombardi, Olaru, Lazar, & Niculescu, 2011). The importance of conservatism in relation to the computational complexity can be measured through the use of set invariance in the control design. The receding horizon control of time-delay systems (Gielen & Lazar, 2011) and the fault-tolerant control of multi-sensor systems (Stankovic, Stoican, Olaru, & Niculescu, 2012) are two examples in this sense. As a contribution, we propose a novel concept of *cyclic invariance*, defined in the current and retarded states concomitantly, with the particularity that the sets need not be identical throughout the delay interval as long as the cyclic inclusion is respected. This leads to a *delay-dependent* construction which covers the existing delay-independent versions and thus obeys the existence conditions defined in such cases (Lombardi et al., 2011), all by relaxing them. Although the cyclic invariance definition is based on Minkowski algebra over the sets, the invariance test can be reduced to an LP (linear programming) feasibility problem (like the classical results for the LTI case in Bitsoris (1988)). Using the same principles, we focus on the possible extensions for additive disturbances and linear feedback design problems.

## 2. Problem formulation and motivation

Consider a delay–difference equation of the form

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time instant $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{Z}_{[0,d]}$ and the initial conditions are given by a sequence $x(-i) \in \mathbb{R}^n$, $i \in \mathbb{Z}_{[0,d]}$. The stability and positive invariance can be studied for the delay–difference equation (1) by considering the equivalent autonomous discrete time system

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k), \tag{2}$$

where $\mathbf{x}(k) = \begin{bmatrix} x(k)^T & x(k-1)^T & \dots & x(k-d)^T \end{bmatrix}^T$ is a vector in $(\mathbb{R}^n)^{d+1}$ and the transition matrix is defined as

$$\Phi = \begin{bmatrix} A_0 & A_1 & \cdots & A_{d-1} & A_d \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}. \tag{3}$$

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*E-mail addresses:* warody.lombardi@lss.supelec.fr (W. Lombardi), sorin.olaru@supelec.fr (S. Olaru), bitsoris@ece.upatras.gr (G. Bitsoris), Silviu.Niculescu@lss.supelec.fr (S.-I. Niculescu).

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The stability of the linear system (2) and the invariant ($\lambda$-contractive) set constructions suffer from the complexity point of view, the dimension of the extended state space — $(\mathbb{R}^n)^{d+1}$ — being directly related to the size of the delay — $d$. We are interested in preserving the analysis of the stability and invariance properties in the original state space $\mathbb{R}^n$ for the dynamical system (1).

**Definition 2.1** (*$\mathcal{D}$-contractiveness, $\mathcal{D}$-invariance*)**.** Let $\lambda \in \mathbb{R}_{[0,1]}$. A set $\mathcal{P} \subset \mathbb{R}^n$ containing the origin is called $\lambda$–$\mathcal{D}$-contractive with respect to (1) if[1]

$$\bigoplus_{i=0}^{d} A_i\mathcal{P} \subseteq \lambda\mathcal{P}. \tag{4}$$

When $\lambda = 1$, $\mathcal{P}$ is said to be a $\mathcal{D}$-invariant set. □

These definitions of $\mathcal{D}$-invariance and $\mathcal{D}$-contractiveness characterizing the behaviour of time-delay systems have to be understood as the set invariance (and contractiveness) concomitantly in the current and delayed states (each being part of $\mathbb{R}^n$). The notion of $\mathcal{D}$-invariance inherits structural conservativeness with respect to the invariant sets corresponding to the extended system (2). The existence of a $\mathcal{D}$-invariant set $\mathcal{P} \subset \mathbb{R}^n$ with respect to (1) implies the invariance of the set $\mathcal{P}^{d+1} \subset (\mathbb{R}^n)^{d+1}$ with respect to (2), but the existence of a non-degenerate invariant set in $(\mathbb{R}^n)^{d+1}$ with respect to (2) does not necessarily imply the existence of a non-degenerate $\mathcal{D}$-invariant set $\mathcal{P} \neq \{0\}$ in $\mathbb{R}^n$ with respect to (1). For convex sets it is interesting to examine whether the $\mathcal{D}$-invariance holds when operations that preserve convexity (like *intersection* or *convex hull*) are used. The following result gives an affirmative answer for the intersection of $\mathcal{D}$-invariant sets.

**Lemma 2.2.** *If the sets $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i \in \mathbb{Z}_{[0,d]}$ are $\mathcal{D}$-invariant with respect to the dynamics (1), then $\bigcap_{i=0}^{d} \mathcal{P}_i$ is a $\mathcal{D}$-invariant set for the same dynamical system.*

## 3. Cyclic $\mathcal{D}$-invariance/contractiveness

A less conservative invariant structure can be obtained by circular shift of a *sequence of "d" convex sets* (or in a general case *of a sequence of q sets*, with $q$ a positive, finite integer). Each one-step circular shift in the sequence brings the previous last set into the first position. Mathematically this insertion in the first position will be allowed by the inclusion properties, as summarized by the next definition.

**Definition 3.1** (*Cyclic $\mathcal{D}$-invariance*)**.** The sequence of sets $\{\mathcal{P}_0, \dots, \mathcal{P}_d\}$ with $0 \in \mathcal{P}_i \subseteq \mathbb{R}^n$ is called cyclically $\mathcal{D}$-invariant with respect to (1) if:

$$\begin{aligned} &A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \oplus \cdots \oplus A_d\mathcal{P}_d \subseteq \mathcal{P}_d;\\ &A_0\mathcal{P}_d \oplus A_1\mathcal{P}_0 \oplus \cdots \oplus A_d\mathcal{P}_{d-1} \subseteq \mathcal{P}_{d-1};\\ &\vdots\\ &A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_2 \oplus \cdots \oplus A_d\mathcal{P}_0 \subseteq \mathcal{P}_0. \end{aligned} \tag{5}$$ □

After $d$ shifts, the sequence comes back to the initial order and thus provides a time-independent formulation of the invariance concept.

Consider the polyhedral sets $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i \in \mathbb{Z}_{[0,d]}$, containing the origin in their interior, and defined by the relations

$$\mathcal{P}_i = \{x \in \mathbb{R}^n \mid F_i x \leq \mathbf{1}\}, \tag{6}$$

where $F_i \in \mathbb{R}^{r\times n}$. For establishing algebraic conditions of cyclic invariance of the sequence $\{\mathcal{P}_0, \dots, \mathcal{P}_d\}$ we define the following matrices:

$$\begin{aligned} &\Psi = diag(F_0, \dots, F_d); \qquad \Theta = diag(A_0, \dots, A_d)\\ &\Gamma_i = \underbrace{\begin{bmatrix} F_i & F_i & \dots & F_i \end{bmatrix}}_{d+1}, \quad i \in \mathbb{Z}_{[0,d]} \end{aligned} \tag{7}$$

and the permutation matrix

$$\Pi = \begin{bmatrix} 0 & 0 & \dots & I_n \\ I_n & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & I_n & 0 \end{bmatrix} \in \mathbb{R}^{(n(d+1))\times(n(d+1))}.$$

**Theorem 3.2.** *The sequence $\{\mathcal{P}_0, \dots, \mathcal{P}_d\}$ of polyhedral sets $\mathcal{P}_i$ is cyclically $\mathcal{D}$-invariant with respect to (1) if and only if there exist matrices $\Omega_i \in \mathbb{R}^{r\times r(d+1)}$, $i \in \mathbb{Z}_{[0,d]}$ with non-negative elements and $\|\Omega_i\|_\infty \leq 1$ such that $\Omega_i\Psi\Pi^i = \Gamma_i\Theta$.*

**Proof.** The series of $d+1$ inclusions defining the $\mathcal{D}$-invariance in (5) can be rewritten explicitly using the half-space representation of the sets $P_i$ in (6) and the permutation matrix

$$\Psi\Pi^i\mathbf{x} \leq \mathbf{1} \Longrightarrow \Gamma_i\Theta\mathbf{x} \leq \mathbf{1}, \quad \forall i \in \mathbb{Z}_{[0,d]}. \tag{8}$$

By virtue of the Extended Farkas Lemma, relation (8) is equivalent to the existence of matrices $\Omega_i$ with non-negative elements satisfying

$$\Omega_i\Psi\Pi^i = \Gamma_i\Theta \quad \text{and} \quad \Omega_i\mathbf{1} \leq \mathbf{1}. \tag{9}$$

Taking into account the elementwise non-negativeness of the matrices $\Omega_i$, the second inequality can be interpreted in terms of the matrix norm as $\|\Omega_i\|_\infty \leq 1$. □

**Remark 3.3.** Theorem 3.2 establishes conditions for the cyclic $\mathcal{D}$-invariance for a sequence of sets of length $d+1$ with respect to the difference equation (1) affected by a delay $d$. Similar conditions can be established for cyclic $\mathcal{D}$-invariance of a sequence of sets $\{\mathcal{P}_0, \dots, \mathcal{P}_q\}$ with $q \neq d$ as long as a series of $\max(d+1, q+1)$ shifted set inclusions similar to those of (5) are verified. For simplicity of notation, the convention of a cyclic sequence of length $d+1$ is preserved throughout the rest of the paper.

## 4. Discussion and related facts

### 4.1. Cyclic $\mathcal{D}$-invariance versus $\mathcal{D}$-invariance

In order to link the basic $\mathcal{D}$-invariance property to the newly introduced cyclic $\mathcal{D}$-invariance, one can observe that for any $\mathcal{D}$-invariant set, there exists a trivial cyclically $\mathcal{D}$-invariant sequence $\{\underbrace{P, \dots, P}_{d+1 \text{ times}}\}$. The next result points toward the converse relationship, which can be summarized as: "cyclic $\mathcal{D}$-invariance induces $\mathcal{D}$-invariance".

**Theorem 4.1.** *Consider the polyhedral sets $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i \in \mathbb{Z}_{[0,d]}$, containing the origin in their interior. If the sequence $\{P_0, \dots, P_d\}$ is cyclically $\mathcal{D}$-invariant with respect to (1) then the intersection*

$$\mathcal{I} = \bigcap_{i=0}^{d} P_i \tag{10}$$

*is $\mathcal{D}$-invariant.*

[1] $\mathcal{A} \oplus \mathcal{B} := \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}$ defines the Minkowski sum of two sets.