Automatica 47 (2011) 192–200

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Averaging techniques without requiring a fast time-varying differential equation☆

Joan Peuteman [a], Dirk Aeyels [b,*]

[a] Katholieke Hogeschool Brugge Oostende, Departement IW&T, lab ECOREA, Zeedijk 101, Oostende B-8400, Belgium
[b] SYSTeMS Research Group, Universiteit Gent, Technologiepark 914, B-9042 Zwijnaarde, Belgium

ARTICLE INFO

Article history:
Available online 3 December 2010

Keywords:
Asymptotic stability
Time-varying
Averaging
Lyapunov

ABSTRACT

An averaging result is presented for local uniform asymptotic stability of nonlinear differential equations without requiring a fast time-varying vectorfield. The nonlinearity plays a crucial role: close to the origin, the trajectories vary slowly compared to the time dependence of the vectorfield. The result generalises averaging results which prove stability properties for systems having a homogeneous vectorfield with positive order. The result is illustrated with several examples.

1. Introduction

It is well known that a solution of a time-varying system $\dot{x}(t) = \varepsilon f(x(t), t)$ may be approximated by the solution of the averaged system $\dot{x}(t) = \varepsilon \bar{f}(x(t))$ on a large time-scale for ε sufficiently small, (Guckenheimer & Holmes, 1983; Khalil, 2002; Verhulst, 2000). The averaging technique also provides a tool to investigate exponential stability of an equilibrium of $\dot{x}(t) = \varepsilon f(x(t), t)$ for ε sufficiently small i.e. exponential stability of the averaged system $\dot{x}(t) = \varepsilon \bar{f}(x(t))$ implies exponential stability of the original time-varying system (Aeyels & Peuteman, 1999; Khalil, 2002). In other words, for ε sufficiently small, exponential stability of the equilibrium point $x = 0$ of $\dot{x}(t) = \bar{f}(x(t))$ implies exponential stability of $x = 0$ of the original fast time-varying system $\dot{x}(t) = f(x(t), t/\epsilon)$. Averaging results are also available for nonsmooth systems when using dither, where solutions of the averaged system (with respect to dither frequency) approximate solutions of the original system (Iannelli, Johansson, Jönsson, & Vasca, 2006).

The averaging concept is useful not only in relation to exponential stability, but also when investigating practical stability properties (Teel, Peuteman, & Aeyels, 1999) and uniform asymptotic stability properties (Peuteman & Aeyels, 1999). In M'closkey (1997) and M'closkey and Murray (1993) the averaging technique is applied to homogeneous systems of order $\tau = 0$. Homogeneous systems with order $\tau > 0$ can also be dealt with: in Peuteman and Aeyels (1999), it is shown that asymptotic stability of the averaged homogeneous system implies local uniform asymptotic stability of the original time-varying homogeneous system without requiring that the original system is fast time-varying.

In this paper, the averaging results discussed in Peuteman and Aeyels (1999) are generalised. Under extra conditions on the differential equation, but without requiring homogeneity of the system, we will show that local asymptotic stability of the averaged system implies local uniform asymptotic stability of the original time varying system. This original time-varying system need not be fast time-varying. Appropriate conditions on the vector field in terms of class K functions imply the local uniform asymptotic stability property without requiring a fast time-varying vectorfield or homogeneity with order $\tau > 0$. What is needed is that, sufficiently close to the equilibrium point $x = 0$, the trajectories are slowly varying compared to the time dependence of the vectorfield. This is usually accomplished through the introduction of a parameter ϵ as indicated above; our main contribution is that in the averaging approach the role played by ϵ may be assumed by the vector field itself.

A number of different examples are included to illustrate our main result and in particular to indicate how it is a generalisation

☆ D.A. has been supported by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Dragan Nesic under the direction of Editor Andrew R. Teel.

* Corresponding author. Tel.: +32 9 264 5656; fax: +32 9 264 5840.
E-mail addresses: joan.peuteman@khbo.be (J. Peuteman), dirk.aeyels@ugent.be (D. Aeyels).

0005-1098/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2010.10.039

of the homogeneous result formulated in Peuteman and Aeyels (1999).

2. The main averaging result

Consider

$$\dot{x}(t) = f(x(t), t) \tag{1}$$

with $f : W \times \mathbf{R}^+ \to \mathbf{R}^n$, W is an open and convex set, $W \subset \mathbf{R}^n$. Let $0 \in W$ and $f(0, t) = 0$ for all $t \in \mathbf{R}^+$. Furthermore, we assume that conditions are imposed on (1) such that existence and uniqueness of its solutions are secured (existence and uniqueness is a standard assumption for all the systems considered in this paper). The system (1) is time-periodic i.e. there exists a $T > 0$ such that for all $x \in W$ and for all $t \in \mathbf{R}^+$,

$$f(x, t) = f(x, t + T). \tag{2}$$

For all $x \in W$, define the averaged system as

$$\dot{x}(t) = \bar{f}(x(t)) \tag{3}$$

where for all $x \in W$,

$$\bar{f}(x) = \frac{1}{T}\int_0^T f(x, t)\mathrm{d}t. \tag{4}$$

We recall when a continuous function is said to belong to class K or to class KL:

The continuous function $\alpha : [0, a) \to \mathbf{R}^+$ (for some $a > 0$) is a class K function if it is strictly increasing and $\alpha(0) = 0$.

The continuous function $\beta : [0, a) \times \mathbf{R}^+ \to \mathbf{R}^+$ (for some $a > 0$) is a class KL function if:

(1) for each fixed s, $\beta(r, s)$ is a class K function in r
(2) for each fixed r, the function $\beta(r, s)$ is decreasing in s and $\beta(r, s) \to 0$ as $s \to +\infty$.

The equilibrium $x = 0$ of (1) is locally uniformly asymptotically stable if there exists a class KL function β and a positive constant c such that $\forall t_0 \geq 0$

$$\|x(t_0)\| < c \Rightarrow \|x(t)\| \leq \beta(x(t_0), t - t_0), \quad \forall t \geq t_0. \tag{5}$$

When for Eq. (3) there exists $t_0 \geq 0$ for which (5) is true, then (5) is true $\forall t_0 \geq 0$: one says that the zero equilibrium of (3) is locally asymptotically stable.

For the $\epsilon - \delta$ definitions of local (uniform) asymptotic stability, the reader is referred to Khalil (2002).

From Lyapunov theory we know that the equilibrium point $x = 0$ of (3) is locally asymptotically stable when there exists an open neighborhood $U \subset W$ of 0 and there exists a Lyapunov function $V : U \to \mathbf{R}$ such that for all $x \in U$:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{6}$$

$$\frac{\partial V}{\partial x}(x)\bar{f}(x) \leq -\alpha_3(\|x\|). \tag{7}$$

Here, $\alpha_1, \alpha_2, \alpha_3 : \mathbf{R}^+ \to \mathbf{R}^+$ are class K functions.

Main theorem. *Assume that the following conditions are satisfied:*

- *the equilibrium point $x = 0$ of the averaged system (3) is locally asymptotically stable, equivalently: (6) and (7) are satisfied,*
- *there exists a class K function $\alpha_4 : \mathbf{R}^+ \to \mathbf{R}^+$ such that for all $x \in U$*

$$\left\|\frac{\partial V}{\partial x}(x)\right\| \leq \alpha_4(\|x\|). \tag{8}$$

- *f is continuously differentiable with respect to x on W for all $t \in \mathbf{R}^+$; furthermore: for all $x \in W$ and for all $t \in \mathbf{R}^+$*

$$\left\|\frac{\partial f}{\partial x}(x, t)\right\| \leq \alpha_5(\|x\|), \tag{9}$$

where, $\alpha_5 : \mathbf{R}^+ \to \mathbf{R}^+$ is a class K function with the additional property that sufficiently close to the origin $\alpha_5(\|x\|(1 + 2T\alpha_5(\|x\|))) \leq 2\alpha_5(\|x\|)$,
- *the function*

$$\alpha_6(\|x\|) \triangleq \frac{\alpha_4(\|x\|)\alpha_5^2(\|x\|)\|x\|}{\alpha_3(\|x\|)} \tag{10}$$

is a class K function ($\alpha_6(0) \triangleq 0$).

Then the equilibrium point $x = 0$ of the original system (1) is locally uniformly asymptotically stable.

The three remarks that follow aim to assess the meaning and significance of condition (9) and (10), and discuss why conditions (9) and (10) are not compatible with linear systems.

Remark 1. In order to conclude that local asymptotic stability of the equilibrium point of the averaged system (3) implies local uniform asymptotic stability of the equilibrium point of the original system (1) without requiring a fast time-varying vectorfield, conditions (9) and (10) are crucial. Condition (9) implies that sufficiently close to the equilibrium point, the trajectories vary slowly in time, compared to the time-dependence of the vectorfield. Condition (9) generalises the homogeneous conditions proposed in Peuteman and Aeyels (1999) where a positive order $\tau > 0$ is required. Condition (10) is more technical and plays a crucial role in part VI of the proof.

Remark 2. We discuss the feasibility of the technical condition that sufficiently close to the origin $\alpha_5(\|x\|(1 + 2T\alpha_5(\|x\|))) \leq 2\alpha_5(\|x\|)$. In case α_5 is continuously differentiable, it is possible to prove this technical condition using the mean value theorem. By continuous differentiability, α_5' is bounded on every arbitrary compact set $[0, r]$. The mean value theorem implies the existence of a $z \in (\|x\|, \|x\|(1 + 2T\alpha_5(\|x\|)))$ such that

$$\alpha_5(\|x\|(1 + 2T\alpha_5(\|x\|))) = \alpha_5(\|x\|) + 2T\alpha_5(\|x\|)\|x\|\alpha_5'(z). \tag{11}$$

Starting with a fixed r, suppose M is an upper bound for $|\alpha_5'|$ on $[0, r]$. Taking $\|x\|$ sufficiently small such that $\|x\|(1+2T\alpha_5(\|x\|)) < r$, one obtains that $|\alpha_5'(z)| \leq M$. With the additional condition that $\|x\| \leq 1/2TM$ such that $2T\alpha_5(\|x\|)\|x\|\ |\alpha_5'(z)| \leq \alpha_5(\|x\|)$, the technical condition is satisfied.

Example 1 (see (82)) and Example 2 (see (94)) also illustrate the feasibility of this technical condition.

Remark 3. For a *linear* system and using a quadratic Lyapunov function V, α_3 is a quadratic function and α_4 is a linear function. Condition (9) is not satisfied since $\partial f/\partial x$ is nonzero at the origin. By replacing α_5 in (9) by a constant bound, also expression (10) does not provide a class K function (one obtains a constant). The main theorem does not prove stability properties: linear systems require fast time-varying vectorfields in order to obtain averaging results (Aeyels & Peuteman, 1999; Khalil, 2002).

Outline of the proof:

First, an appropriate change of variables (21) is defined. This leads to the system (26) in y which is equivalent with the original system (1) in x. Since (26) may be seen as a perturbation of the averaged system (3), the Lyapunov function V is invoked to prove local uniform asymptotic stability of the equilibrium point of (26). Formulating this stability property in terms of class KL