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Reciprocally convex approach to stability of systems with time-varying delays[☆]

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ABSTRACT

Whereas the upper bound lemma for matrix cross-product, introduced by Park (1999) and modified by Moon, Park, Kwon, and Lee (2001), plays a key role in guiding various delay-dependent criteria for delayed systems, the Jensen inequality has become an alternative as a way of reducing the number of decision variables. It directly relaxes the integral term of quadratic quantities into the quadratic term of the integral quantities, resulting in a linear combination of positive functions weighted by the inverses of convex parameters. This paper suggests the lower bound lemma for such a combination, which achieves performance behavior identical to approaches based on the integral inequality lemma but with much less decision variables, comparable to those based on the Jensen inequality lemma.

1. Introduction

In the field of stability analysis and control design, tighter upper bounds of various functions have been pursued: affine functions of polytopic-type uncertain systems, quadratic functions of T–S fuzzy control systems, and especially a special type of function combinations in delayed systems (Jiang & Han, 2005; Shao, 2009) which is the focus of our discussion.

This particular featured function originates from the relaxation based on the Jensen inequality lemma (Gu, Kharitonov, & Chen, 2003) in the delayed systems. Initially, the integral inequality lemma for matrix cross-products (Ko & Park, 2009; Moon et al., 2001; Park, 1999) has played a key role in guiding various delay-dependent criteria with the choice of the Lyapunov–Krasovskii functional introduced in Fridman and Shaked (2003). However, to fully relax the matrix cross-products, it has to introduce slightly excessive free weighting matrices. As a way of reducing the number of decision variables, at the sacrifice of conservatism, relaxations based on the Jensen inequality lemma (Gu et al., 2003) have been attracted, recently (Wu, Feng, & He, 2009; Zhu, Yang, Li, Lin, & Guo, 2009). It directly relaxes an integral term of quadratic quantities into a quadratic term of integral quantities. Such relaxed quadratic terms appear as a linear combination of positive functions weighted by the inverses of convex parameters. As concerned about it, Shao (2009) has achieved an excellent work of reducing the conservativeness. The basic idea is to approximate the integral terms of quadratic quantities into a convex combination of quadratic terms of the integral quantities. However, owing to the inversely weighted nature of coefficients in the Jensen inequality approach, Shao (2009) has to introduce an approximation on the difference between delay bounds, $\int_{t-h_2}^{t-h(t)}(h_2-h_1)f(\alpha)\mathrm{d}\alpha \geq \int_{t-h_2}^{t-h(t)}(h_2-h(t))f(\alpha)\mathrm{d}\alpha$, $h_1 \leq h(t) \leq h_2$, in the middle stage of the derivation.

This paper suggests a lower bound lemma for such a linear combination of positive functions with inverses of convex parameters as the coefficients. Based on the lemma, we develop a stability criterion that directly handles the inversely weighted convex combination of quadratic terms of integral quantities, which achieves performance behavior identical to approaches based on the integral inequality lemma but with much less decision variables, comparable to those based on the Jensen inequality lemma.

The paper is organized as follows. Section 2 provides a new lower bound lemma for a weighted linear combination of positive functions over the inverses of convex parameters. Based on this lemma, it considers a stability criterion to show how to handle

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the double integral terms of a Lyapunov–Krasovskii functional for delayed systems without introducing a convexizing anti-inverse process. Section 3 will show simple examples for the verification of the criterion.

2. Main results

In this paper, we concern a special type of function combinations, *i.e.* a linear combination of positive functions with inverses of convex parameters as the coefficients, which is defined below.

2.1. Reciprocally convex combination

Definition 1. Let $\Phi_1, \Phi_2, \ldots, \Phi_N : \mathbf{R}^m \mapsto \mathbf{R}^n$ be a given finite number of functions such that they have positive values in an open subset $\mathbf{D}$ of $\mathbf{R}^m$. Then, a *reciprocally convex combination* of these functions over $\mathbf{D}$ is a function of the form

$$\frac{1}{\alpha_1}\Phi_1 + \frac{1}{\alpha_2}\Phi_2 + \cdots + \frac{1}{\alpha_N}\Phi_N : \mathbf{D} \mapsto \mathbf{R}^n, \tag{1}$$

where the real numbers α_i satisfy $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

The following theorem suggests a lower bound for a reciprocally convex combination of scalar positive functions $\Phi_i = f_i$.

2.2. Lower bounds theorem

Theorem 1. *Let $f_1, f_2, \ldots, f_N : \mathbf{R}^m \mapsto \mathbf{R}$ have positive values in an open subset $\mathbf{D}$ of $\mathbf{R}^m$. Then, the reciprocally convex combination of f_i over $\mathbf{D}$ satisfies*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t) \tag{2}$$

subject to

$$\left\{ g_{i,j} : \mathbf{R}^m \mapsto \mathbf{R}, g_{j,i}(t) \triangleq g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}. \tag{3}$$

Proof. The constraint in (3) implies

$$\begin{bmatrix} \sqrt{\frac{\alpha_j}{\alpha_i}} \\ -\sqrt{\frac{\alpha_i}{\alpha_j}} \end{bmatrix}^T \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\alpha_j}{\alpha_i}} \\ -\sqrt{\frac{\alpha_i}{\alpha_j}} \end{bmatrix} \geq 0. \tag{4}$$

Then, we have

$$\sum_i \frac{1}{\alpha_i} f_i(t) = \sum_{i,j} \frac{\alpha_j}{\alpha_i} f_i(t) \tag{5}$$

$$= \sum_i f_i(t) + \frac{1}{2}\sum_{i\neq j}\left(\frac{\alpha_j}{\alpha_i} f_i(t) + \frac{\alpha_i}{\alpha_j} f_j(t)\right) \tag{6}$$

$$\geq \sum_i f_i(t) + \sum_{i\neq j} g_{i,j}(t). \tag{7}$$

Note that the equality holds for

$$\alpha_i = \frac{\sqrt{f_i(t)}}{\sum_j \sqrt{f_j(t)}}, \qquad g_{i,j}(t) = \sqrt{f_i(t) f_j(t)}, \tag{8}$$

which completes the proof. □

This theorem can be applied to handle the double integral terms of the Lyapunov–Krasovskii functional for delayed systems, which is the focus of the forthcoming section.

2.3. Application to delayed systems

Let us consider the following delayed system:

$$\dot{x}(t) = Ax(t) + A_h x(t - h(t)), \quad t \geq 0, \tag{9}$$

$$x(t) = \phi(t), \quad -h_2 \leq t \leq 0, \tag{10}$$

where $0 \leq h_1 \leq h(t) \leq h_2$, $h_{12} \triangleq h_2 - h_1$ and $\phi(t) \in \mathcal{C}^1(h_2)$, the set of continuously differentiable functions in the domain $[-2h_2, 0]$. Let us define $\chi(t) \triangleq [x^T(t)\, x^T(t-h(t))\, x^T(t-h_1)\, x^T(t-h_2)]^T$ and the corresponding block entry matrices as

$$e_1 \triangleq \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^T, \qquad e_2 \triangleq \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}^T,$$

$$e_3 \triangleq \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}^T,$$

$$e_4 \triangleq \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^T, \qquad e_5 \triangleq (Ae_1^T + A_h e_2^T)^T, \tag{11}$$

such that system (9) can be written as $\dot{x}(t) = e_5^T \chi(t)$.

Consider the following Lyapunov–Krasovskii functional:

$$V(t) \triangleq V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{12}$$

$$V_1(t) = x^T(t) P x(t), \quad P > 0, \tag{13}$$

$$V_2(t) = \int_{t-h_1}^{t} x^T(\alpha) Q_1 x(\alpha) \mathrm{d}\alpha, \quad Q_1 > 0, \tag{14}$$

$$V_3(t) = \int_{t-h_2}^{t} x^T(\alpha) Q_2 x(\alpha) \mathrm{d}\alpha, \quad Q_2 > 0, \tag{15}$$

$$V_4(t) = h_1 \int_{-h_1}^{0} \int_{t+\alpha}^{t} \dot{x}^T(\beta) R_1 \dot{x}(\beta) \mathrm{d}\beta \mathrm{d}\alpha, \quad R_1 > 0, \tag{16}$$

$$V_5(t) = h_{12} \int_{-h_2}^{-h_1} \int_{t+\alpha}^{t} \dot{x}^T(\beta) R_2 \dot{x}(\beta) \mathrm{d}\beta \mathrm{d}\alpha, \quad R_2 > 0. \tag{17}$$

Then, the time derivative of $V(t)$ becomes

$$\dot{V}_1(t) = 2\dot{x}^T(t) P x(t) = 2\chi^T(t) e_5 P e_1^T \chi(t), \tag{18}$$

$$\dot{V}_2(t) = \chi^T(t)\{e_1 Q_1 e_1^T - e_3 Q_1 e_3^T\}\chi(t), \tag{19}$$

$$\dot{V}_3(t) = \chi^T(t)\{e_1 Q_2 e_1^T - e_4 Q_2 e_4^T\}\chi(t), \tag{20}$$

$$\dot{V}_4(t) = h_1^2 \chi^T(t) e_5 R_1 e_5^T \chi(t) - h_1 \int_{t-h_1}^{t} \dot{x}^T(\alpha) R_1 \dot{x}(\alpha) \mathrm{d}\alpha, \tag{21}$$

$$\begin{aligned} \dot{V}_5(t) &= h_{12}^2 \chi^T(t) e_5 R_2 e_5^T \chi(t) - h_{12} \int_{t-h_2}^{t-h_1} \dot{x}^T(\alpha) R_2 \dot{x}(\alpha) \mathrm{d}\alpha, \\ &= h_{12}^2 \chi^T(t) e_5 R_2 e_5^T \chi(t) - h_{12} \int_{t-h(t)}^{t-h_1} \dot{x}^T(\alpha) R_2 \dot{x}(\alpha) \mathrm{d}\alpha \\ &\quad - h_{12} \int_{t-h_2}^{t-h(t)} \dot{x}^T(\alpha) R_2 \dot{x}(\alpha) \mathrm{d}\alpha. \end{aligned} \tag{22}$$

Define the non-integral terms of $\dot{V}(t)$ as $\chi^T(t)\Pi\chi(t)$, where

$$\begin{aligned} \Pi \triangleq{}& e_5 P e_1^T + e_1 P e_5^T + e_1 Q_1 e_1^T - e_3 Q_1 e_3^T + e_1 Q_2 e_1^T \\ & - e_4 Q_2 e_4^T + h_1^2 e_5 R_1 e_5^T + h_{12}^2 e_5 R_2 e_5^T. \end{aligned} \tag{23}$$

To handle the integral terms in $\dot{V}_4(t)$ and $\dot{V}_5(t)$, the integral inequality lemma (Moon et al., 2001; Park, 1999) has been generally adopted. However to reduce the number of free weighting matrices, at the sacrifice of conservativeness, relaxations based on the Jensen inequality lemma (Gu et al., 2003) have become alternatives. Here we introduce our main result, which shows that Theorem 1 enables us to achieve both benefits: performance behavior identical to approaches based on the integral inequality lemma but with the number of decision variables, comparable to those based on the Jensen inequality lemma.