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Implicit discrete-time systems and accessibility ☆

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ABSTRACT

An intrinsic description for dynamic systems, whose evolution along discrete time is governed by (nonlinear) implicit difference equations in one independent variable and zero-order (algebraic) equations, is presented by means of differential geometrical methods, where systems are associated with appropriate geometric objects reflecting their dynamics. Dynamic systems given in implicit form have the peculiarity that they may contain so-called hidden restrictions. A normal form is presented which is characterized by the circumstances that there are no further restrictions. In addition, it is illustrated that such a normal form allows for an equivalent system representation in explicit form. Based on the geometric picture of (implicit) discrete-time systems the qualitative property of accessibility along a fixed trajectory is discussed. By applying symmetry groups of discrete-time systems and studying invariants of these groups a formal approach is provided that allows us to gather local accessibility criteria successively, which can be tested by computer algebra. Several examples illustrate the results.

1. Introduction

In modeling physical processes systems, which evolve along discrete time, arise quite naturally. Such discrete-time procedures are found in multiple problems of finance and economics for instance. Discrete-time systems even appear by the (semi-) discretization of lumped and distributed parameter systems; in control theory it is well established that this class of dynamic systems is of particular interest regarding the digital implementation of control laws.

In the present work it is supposed that a discrete-time dynamic process is described by a set of n_e implicit difference equations

$$0 = f^{\alpha_e}(k, z(k), z(k+1)), \quad \alpha_e = 1, \dots, n_e \tag{1a}$$

and n_y output functions

$$y^{\alpha_y}(k) = c^{\alpha_y}(k, z(k)), \quad \alpha_y = 1, \dots, n_y, \tag{1b}$$

where the independent variable $k \in \mathbb{Z}$ denotes the discrete time steps and a solution sequence is indicated by $z^{\alpha_z}(k)$, $\alpha_z = 1, \dots, n_z$. An intrinsic formulation is encouraged by identifying implicit systems with a geometric object; namely, the system equations describe a submanifold of some larger space. Systems given in implicit form have the peculiarity that they may contain so-called hidden restrictions. In this context a normal form is proposed such that a representation of a discrete-time system without additional (hidden) equations is obtained, which may describe a smaller submanifold; hence, it contains all possible solutions of the system. Moreover, such a geometric picture offers the possibility to apply well appreciated tools for the analysis of nonlinear systems, see, e.g., Grizzle (1985), Monaco and Normand-Cyrot (1984), Nijmeijer and van der Schaft (1991), and references therein.

A main issue of this paper is to emphasize that differential geometric methods are appropriate for a study of implicit discrete-time dynamic systems by taking the derived intrinsic geometric description into account. In order to motivate for a system-theoretical analysis we confine to the investigation of the accessibility property. A crucial observation is that the formal Lie group approach for the analysis of continuous-time dynamic systems is applicable in discrete time, see, e.g., Holl (2005), Rieger, Schlacher, and Holl (2008) and Schlacher, Kugi, and Zehetleitner (2002), where local criteria can be derived via an infinitesimal principle of invariance without utilizing the solution of the dynamic system explicitly. The obtained tests can be checked by computer algebra at least for polynomial systems (with Groebner bases, see, e.g., Cox, Little, & O'Shea, 1992).

The accessibility property of (explicit) discrete-time systems is well studied in the literature, see, e.g., Albertini and Sontag (1993), Aranda-Bricaire, Kotta, and Moog (1996), Barbot (1990), Fliegner (1995), Fliess and Normand-Cyrot (1981), Halás, Kotta, Li, Wang, and Yuan (2009), Jakubczyk and Sontag (1990), Kotta (2005), Simes (1996), Wirth (1998), Zhang and Zheng (2004) and references therein. In contrast to the proposed approach we refer to, e.g., Albertini and Sontag (1993), Aranda-Bricaire

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et al. (1996), Fliess and Normand-Cyrot (1981) and Jakubczyk and Sontag (1990), where also geometric methods are applied and criteria for accessibility with respect to explicit (invertible) discrete-time systems in terms of appropriate distributions and codistributions are derived, respectively, hence, using a different notion of accessibility. Here, accessibility is studied along a trajectory (with given input) and for a fixed number of time steps. Whereas in, e.g., Albertini and Sontag (1993), Aranda-Bricaire et al. (1996) and Jakubczyk and Sontag (1990) all possible inputs and an arbitrary number of time steps are considered and, thus, in the context of the definition, used in this contribution, the question is answered whether there exists a trajectory such that the system is accessible. To the knowledge of the authors there are not many publications focusing on implicit discrete-time nonlinear control systems, see, e.g., Fliegner, Kotta, and Nijmeijer (1996), Fliegner (1995), Simes (1996), Simes and Nijmeijer (1995) and references therein.

This contribution is organized as follows. In Section 2 the intrinsic description of implicit discrete-time dynamic systems is presented. In Section 3 a normal form for implicit systems is derived and it is shown that it is equivalent to an explicit one at least locally. Based on the intrinsic picture of systems the accessibility problem for discrete-time systems is treated in Section 4. By the consideration of invariants of (local) transformation groups acting on the solution of the dynamic system (local) criteria for the accessibility along a fixed trajectory are obtained. In order to illustrate the straightforward applicability of the developed theory, the presented methods are applied to several examples. The contribution finishes with some conclusions.

2. A geometric description of implicit discrete-time systems

In system theory dynamic systems are identified with geometric objects in order to obtain a coordinate-free representation, see, e.g., Isidori (1995). Thus, it is the intention to consider geometric objects together with diffeomorphic changes of local coordinates equipped with a structure, which reflects the dynamics of discrete-time systems. The construction of an intrinsic representative relies here on the concepts of manifolds and bundles, where the interested reader is referred to, e.g., Saunders (1989) and Spivak (1979) for an introduction and much more detail.

2.1. Dynamic systems in implicit form

For the investigation of implicit systems we utilize a system representation by means of a set of general coordinates resp. so-called descriptor variables, see, e.g., Luenberger (1977) for a similar approach, such that there is no distinction of state and input variables *a priori*. Let us introduce the dependent coordinates z_k as descriptor variables as well as k as an independent coordinate with[1] $z_k = z(k)$ and $z_{k+1} = z(k+1)$, where $z(k)$ is a solution sequence of the system (1). Then, an implicit system can be rewritten in the form

$$0 = f^{\alpha_e}(k, z_k, z_{k+1}), \quad \alpha_e = 1, \ldots, n_e,$$
$$y_k^{\alpha_y} = c^{\alpha_y}(k, z_k), \quad \alpha_y = 1, \ldots, n_y. \tag{2}$$

If the initial condition $z(k_0) = z_{k_0}$ is considered, the solution sequence $z(k)$ can be determined for $k \geq k_0$, which is not unique in general and may even not exist. It is worth mentioning that the existence and uniqueness of solutions can be related to the so-called regularization problem, see, e.g., Xiaoping and Celikovsky (1997) for continuous-time affine nonlinear singular control systems.

In order to point out the geometric picture of an implicit system (2) the bundle $(\mathcal{E}_\mathbb{Z}, \pi_\mathbb{Z}, \mathcal{B})$ is introduced. Basically, k is the coordinate on the (discrete) base manifold $\mathcal{B}$, which is isomorphic to the set of integers $\mathbb{Z}$, see, e.g., Munkres (1991), and (k, z_k) with $z_k = (z_k^{\alpha_z})$, $\alpha_z = 1, \ldots, n_z$ are the (local) coordinates on the total manifold $\mathcal{E}_\mathbb{Z}$. Then, the projection becomes $\pi_\mathbb{Z} : \mathcal{E}_\mathbb{Z} \to \mathcal{B}; (k, z_k) \mapsto (k)$. The set of sections $\gamma : \mathcal{B} \to \mathcal{E}_\mathbb{Z}$ with $\gamma(k) = (k, z^{\alpha_z}(k)) = (k, \gamma^{\alpha_z}(k))$ is denoted $\Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$. In order to avoid any mathematical irregularities from now on it is assumed that all manifolds are smooth manifolds and the functions f and c depend smoothly on their arguments for fixed k.[2] If the latter assumption do not hold, the presented methods do not lose their validity; hence, often many distinctions of cases can be avoided.

Let $s_k^n(\gamma)$ denote the equivalence class of all sections $\bar{\gamma} \in \Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$ at $k \in \mathcal{B}$ such that $\bar{\gamma}(k) = \gamma(k)$ as well as $\sigma^i(\bar{\gamma}(k)) = \sigma^i(\gamma(k))$, $i = 1, \ldots, n$ is satisfied, where $\sigma^i(\cdot)$ denotes the i-times application of the (forward) shift operator, $\sigma(\gamma(k)) = \gamma(k+1)$. Then, the set of equivalence classes $s_k^n(\gamma)$ can be endowed with the structure of a manifold $S^n(\mathcal{E}_\mathbb{Z}) = \{s_k^n(\gamma) \mid k \in \mathcal{B}, \gamma : \mathcal{B} \to \mathcal{E}_\mathbb{Z}\}$, equipped with the adapted coordinates $(k, z_k, z_{k+1}, \ldots, z_{k+n})$. The prolongation of a section $\gamma \in \Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$ to $\Gamma(\mathcal{B}, S^n(\mathcal{E}_\mathbb{Z}))$ is denoted $s^n(\gamma)$ with $s^n(\gamma)(k) = (k, \gamma(k), \gamma(k+1), \ldots, \gamma(k+n))$.

Example 1. The sections $\gamma, \bar{\gamma} \in \Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$ with $n_z = 1$ and $\gamma(k) = (k, k+2)$, $\bar{\gamma}(k) = (k, k^2+2)$ belong to the same equivalence class $s_0^1(\gamma)$ at $k = 0$ because $\gamma(0) = \bar{\gamma}(0) = 2$ and $\gamma(1) = \bar{\gamma}(1) = 3$.

Remark 2. The manifolds $S^n(\mathcal{E}_\mathbb{Z})$ represent the counterpart of the well-established jet manifolds, see, e.g., Saunders (1989), used for continuous-time systems, which are defined under the assumption of a smooth base manifold $\mathcal{B}$, isomorphic to $\mathbb{R}$, as the equivalence class of all sections $\bar{\gamma} \in \Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$ satisfying $\bar{\gamma}(t) = \gamma(t)$ as well as $\partial_t^i \bar{\gamma}(t) = \partial_t^i \gamma(t)$, $i = 1, \ldots, n$, where ∂_t^i indicates the ith partial derivative with respect to the coordinate t on $\mathcal{B}$.

According to the construction of the manifolds $S^n(\mathcal{E}_\mathbb{Z})$ the triple $(S^n(\mathcal{E}_\mathbb{Z}), \pi_0^n, \mathcal{E}_\mathbb{Z})$ represents a bundle with the projection $\pi_0^n : S^n(\mathcal{E}_\mathbb{Z}) \to \mathcal{E}_\mathbb{Z}; (k, z_k, z_{k+1}, \ldots, z_{k+n}) \mapsto (k, z_k)$. A diffeomorphic change of coordinates, which preserves the bundle structure, reads as[3]

$$\bar{z}_{\bar{k}}^{\alpha_z} = \psi_z^{\alpha_z}(k, z_k), \quad \alpha_z = 1, \ldots, n_z,$$
$$\bar{z}_{\bar{k}+i}^{\alpha_z} = \psi_z^{\alpha_z}(k+i, z_{k+i}), \quad i = 1, \ldots, n. \tag{3}$$

A geometric picture of an implicit discrete-time system follows by the assumption that the system equations[4] $f_k = f(k, z_k, z_{k+1}) = 0$ describe a (locally) regular submanifold $\mathcal{M}_k \subset S^1(\mathcal{E}_\mathbb{Z})$ for any $k \geq k_0$ with constant dimension, see, e.g., Giachetta, Mangiarotti, and Sardanashvily (1997), Saunders (1989) and references cited therein. In addition, we have the map $c : \mathcal{E}_\mathbb{Z} \to \mathcal{Y}$, $c^{\alpha_y}(k, z_k) \in C^\infty(\mathcal{E}_\mathbb{Z})$ with $y_k \in \mathcal{Y}$. A section $\gamma \in \Gamma(\mathcal{B}, \mathcal{E}_\mathbb{Z})$ with $\gamma : k \mapsto \gamma(k) = (k, z^{\alpha_z}(k)) = (k, \gamma^{\alpha_z}(k))$, which satisfies the equations

$$f^{\alpha_e} \circ s^1(\gamma)(k) = f^{\alpha_e}(k, z(k), z(k+1)) = 0 \tag{4}$$

is called a solution of the implicit discrete-time system with the initial condition $\gamma(k_0) = (k_0, z(k_0)) = (k_0, z_{k_0})$.

1 Note that z_k is a coordinate and $z(k)$ is the kth element of a sequence z.

2 For the succeeding investigations actually only C^1-manifolds and functions f, c of class C^1 are required.

3 For an abbreviation $\bar{k} = \psi_k(k)$ is neglected where it is assumed that the transformation ψ_k retains the ordering such that for any $k_2 > k_1$, $\psi_k(k_2) > \psi_k(k_1)$ is implied.

4 The subscript k denotes the evaluation of an expression for fixed k.