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Brief paper

Distributed consensus over digital networks with limited bandwidth and time-varying topologies☆

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ABSTRACT

In this paper, we consider discrete-time distributed average-consensus with limited communication data-rate and time-varying communication topologies. We design a distributed encoding-decoding scheme based on quantization of scaled innovations and a control protocol based on a symmetric compensation method. We develop an adaptive scheme to select the numbers of quantization levels according to whether the associated channel is active or not. We prove that if the network is jointly connected, then under the protocol designed, average-consensus can be asymptotically achieved, and the convergence rate is quantified. Especially, if the duration of any link failure in the network is bounded, then the control gain and the scaling function can be selected properly such that 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

1. Introduction

In recent years, distributed coordination over multi-agent networks has become one of the most dynamic directions of networked control systems and complexity science. When digital communications are adopted, due to the finite channel capacity, only a finite number of bits of information can be exchanged between neighbors at each time step. The communication between agents is a combined process of encoding, transmission, receiving and decoding. Similar to the traditional single-agent control theory, the research on multi-agent coordination started initially without considering the communication constraint, then with the deepening of the study, various aspects of communication networks were gradually paid attention to by researchers. Recently, quantized consensus or consensus with quantized communications has drawn the attention of more and more researchers (Kashyap, Basar, & Srikant, 2007; Carli, Fagnani, Frasca, & Zampieri, 2007; Frasca, Carli, Fagnani, & Zampieri, 2009; Carli, Bullo, & Zampieri, 2010a; Carli, Fagnani, Frasca, & Zampieri, 2010b; Li, Fu, Xie, & Zhang, 2011).

Most of the existing works on distributed consensus with quantized communications assume time-invariant communication topologies. It is well-known that in multi-agent networks the communication topology is often time-varying due to many reasons such as link failure or the change of environment. The change of network topologies due to link failures is a kind of passive switching. A well distributed consensus algorithm should be robust against this kind of switching. In some other cases, the network topology may be changed on purpose, for example, the network switches among different modes according to high-level commands for performance optimization of the whole system. Due to the limited bandwidth, distributed consensus with quantized communications and time-varying topologies is of significance from both theoretic and engineering points of view. Dimarogonas and Johansson (2008) considered distributed consensus based on quantized relative state information with static infinite-level uniform and logarithmic quantizers and proved that if the communication graph remains a tree for all consecutive time intervals between switching points then consensus can be achieved provided the quantization density is sufficiently high. Nedić, Olshevsky, Ozdaglar, and Tsitsiklis (2009) considered quantized average-consensus with time-varying topology and static infinite-level uniform quantizers. They proved that if the

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communication graph is jointly connected, then approximate average-consensus can be achieved. Carli et al. (2010b) and Lavaei and Murray (2009a,b) considered a random gossip algorithm with quantized communications based on an infinite-level uniform quantizer. Carli et al. (2010b) proved that if the edges selected form a connected graph, then the states of agents converge to an approximate average of the initial values up to the size of the quantization interval. More performance analysis can be found in Lavaei and Murray (2009a,b). Kar and Moura (2009) considered quantized average-consensus with random link failures by static infinite-level and finite-level uniform quantizers. They added a random dither before quantization to make the quantization error a "white" noise and proved that if the sequence of Laplacian matrices is i.i.d and the mean graph is connected all the time, then all agents' states eventually enter into a small neighborhood of the average of the initial states with high probability. Note that most of the above literature concentrates on static infinite-level quantizers and the steady-state error is not zero.

In this paper, we consider discrete-time average-consensus with limited communication bandwidth. The agents have real-valued states and communicate with each other through undirected digital networks. The network topology can be time-varying. There are two fundamental difficulties. One is that the communication channels between agents have only limited capacity, so only finite-level quantizers can be used, which may result in unbounded quantization errors. The other is that the neighbors of a given agent may change with time, which may result in a mismatch between the encoder and decoder designed for the fixed topology case (Li et al., 2011). To overcome these difficulties, we design a distributed encoding-decoding scheme based on quantization of scaled innovations, where a scaling function is used to avoid the saturation of the finite-level quantizers. Here, unlike the fixed topology case, each communication channel has its own encoder and decoder. The number of quantization levels of each quantizer depends on the status of the associated communication channel. We propose a control protocol based on symmetric compensation and design an adaptive scheme to select the numbers of quantization levels. The number of quantization levels of each quantizer is tuned on-line according to whether the associated channel is active or not at the last step. By using this method, we prove that if the network is jointly connected, then under the protocol designed, average-consensus can be asymptotically achieved without steady-state error, which means that average-consensus can be achieved with arbitrary precision as time goes on. We show that the convergence rate is no lower than that of the scaling function. Especially, if the duration of any link failures in the network is bounded, then the control gain and the scaling function can be selected properly such that 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

The remainder of this paper is organized as follows. In Section 2, we present the model of the network and formulate the problem to be investigated. In Section 3, we propose an encoding-decoding scheme and a control protocol. In Section 4, we consider how to select the control parameters to ensure asymptotic average-consensus and give the convergence rate of the closed-loop system. In Section 5, we give a numerical example to demonstrate the validity of our protocol. In Section 6, some concluding remarks and future research topics are provided.

The following notation will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones. I denotes the identity matrix with an appropriate size. For a given set $\mathcal{S}$, the number of its elements is denoted by $|\mathcal{S}|$. For a given vector or matrix A, its transpose is denoted by A^T, its ∞-norm is denoted by $\|A\|_\infty$, and its Euclidean norm is denoted by $\|A\|$. For a given positive number x, the logarithm of x with base 2 is denoted by $\log_2(x)$; the maximum integer less than or equal to x is denoted by $\lfloor x \rfloor$, and the minimum integer greater than or equal to x is denoted by $\lceil x \rceil$. For two real symmetric matrices A and B, we denote $A \le B$ if $B - A$ is positive semi-definite.

2. Model description and problem formulation

We consider a network of N agents with the dynamics

$$x_i(t+1) = x_i(t) + u_i(t), \quad t = 0, 1, \ldots, \ i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are, respectively, the state and input of the ith agent. The communications between agents at time t are represented by the undirected graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{A}(t)\}$, $t = 1, 2 \ldots$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of agents, and $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{G}(t)$ with element $a_{ij}(t) = 1$ or 0 indicating whether or not there is an active communication channel from j to i.

Since the communication graphs are undirected, $\mathcal{A}(t)$ is a symmetric matrix. The neighborhood of agent i at time t is defined as $N_i(t) = \{j \in \mathcal{V} \mid a_{ij}(t) = 1\}$. The cardinal number of $N_i(t)$ is called the degree of i at time t and is denoted by $d_i(t)$. Denote $\mathcal{N}_i = \bigcap_{t=1}^{\infty} \bigcup_{k=t}^{\infty} N_i(k)$.

The Laplacian matrix of $\mathcal{G}(t)$ is defined as $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where $\mathcal{D}(t) = \mathrm{diag}(d_1(t), \ldots, d_N(t))$. We represent the communication channel from j to i by pair (j, i). A sequence of active communication channels $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ is called a path from agent i_1 to agent i_k. The graph $\mathcal{G}(t)$ is called a connected graph if for any $i, j \in \mathcal{V}$, there is a path from i to j.

For the sequence $\{\mathcal{G}(t), t = 1, 2, \ldots\}$, we have the following assumption.

(A1) $\quad \mathcal{N}_i = \bigcup_{t=1}^{\infty} N_i(t)$.

Remark 1. Observe that $\mathcal{N}_i$ represents the set of agents which are the neighbors of agent i for an infinite number of times. Assumption (A1) means that if agent j is a neighbor of agent i for some time, then it is also a neighbor of i for an infinite number of times. Note that channels which only exist for a finite period will have no essential impact on the protocol design and closed-loop analysis.

In this paper, we aim to design an encoding-decoding scheme using a finite number of bits of data and an interaction protocol to achieve asymptotic average-consensus.

3. Protocol design

We assume that only symbolic data can be exchanged between agents. For each communication channel, the state value of the sender is firstly encoded to symbolic data and then transmitted, after the data is received, a decoder is used by the receiver to get an estimate of the sender's state. For agent i and agent j, where $j \in \mathcal{N}_i$, $i = 1, 2, \ldots, N$, the encoder ϕ_{ij} associated with i for channel (i, j) is designed as

$$\begin{cases} \xi_{ij}(0) = 0, \\ \xi_{ij}(t) = g(t-1)a_{ij}(t)s_{ij}(t) + \xi_{ij}(t-1), \\ s_{ij}(t) = q_t^{ij}\left(\dfrac{x_i(t) - \xi_{ij}(t-1)}{g(t-1)}\right), \quad t = 1, 2, \ldots, \end{cases} \tag{2}$$

where $\xi_{ij}(t)$ is the internal state of ϕ_{ij}, $x_i(t)$ and $s_{ij}(t)$ are the input and the output of ϕ_{ij}, respectively. Here, $q_t^{ij}(\cdot)$ is a quantizer, and $g(t) > 0$ is a scaling function. For agent $l \notin \mathcal{N}_i$, the channel (i, l) will never be active, and it is unnecessary to design an encoder for (i, l). Therefore, there are $|\mathcal{N}_i|$ encoders associated with agent i in total.