Available online at www.sciencedirect.com
SCIENCE DIRECT®
Automatica 41 (2005) 359–376
automatica
www.elsevier.com/locate/automatica

Asymptotic properties of subspace estimators☆

Dietmar Bauer*

Institute for Mathematical Methods in Economics, TU Wien, Argentinierstr. 8, A-1040 Wien, Austria

Received 27 January 2004; received in revised form 22 July 2004; accepted 12 November 2004
Available online 18 January 2005

Abstract

Since the proposal of subspace methods in the 1980s and early 1990s substantial efforts have been made in the analysis of the statistical properties of the algorithms. This paper surveys the literature on the asymptotic properties of particular subspace methods used for linear, dynamic, time invariant, discrete time systems. The goals of this paper are threefold: First this survey tries to present the most relevant results on the asymptotic properties of estimators obtained using subspace methods. Secondly the main methods and tools that have been used in the derivation of these results are presented to make the literature more accessible. Thirdly main unsolved questions and rewarding research topics are identified some of which can be attacked using the toolbox discussed in the paper.

Keywords: Linear systems; Discrete time systems; Subspace methods; Asymptotic properties

1. Introduction

Since the seminal survey of Viberg (1995) on subspace methods for the identification of linear dynamic models the literature on subspace methods has grown substantially. The development can be very coarsely divided into four directions of research (without any particular ordering)

(1) Extensions of the basic underlying idea to different model classes.
(2) Development of algorithms for online applicability and monitoring purposes.
(3) Application of the methods.
(4) Analysis of the asymptotic properties of the estimators.

Each of these topics would warrant a separate survey paper. In this paper I will focus on the fourth topic. In particular the scope is constrained to the case of linear, dynamical, time invariant, discrete time systems since this to the best of my knowledge is the only class of models where the asymptotic properties of the algorithms have been investigated to a certain extent.

In order to survey asymptotic properties of subspace methods it is necessary to make the definition of the methods under investigation more concrete. The term *subspace methods* unfortunately is used for a wide variety of quite different algorithms. All subspace methods are formulated in the state space framework for modelling the dependence of some measured output process $(y_t)_{t\in\mathbb{Z}}$ where $y_t \in \mathbb{R}^s$, on an observed input process $(u_t)_{t\in\mathbb{Z}}$ where $u_t \in \mathbb{R}^m$, and lagged values of the inputs and outputs. A common feature of all model classes for which subspace methods have been developed is that the vector of stacked outputs $y_{t+j},\ j = 0, 1, \ldots, f-1$, can be additively decomposed into a linear function of the n-dimensional state x_t, a possibly nonlinear function of $u_{t+j},\ j = 0, \ldots, f-1$ and a noise component which according to the models is uncorrelated with the remaining two terms. This last term equals the prediction errors of a prediction of y_{t+j}, $j \geqslant 0$ based on $(u_t)_{t\in\mathbb{Z}}$ and $y_s, s < t$, and the sum of the first two the mean square predictions. The term 'subspace' then is motivated from the fact that for $fs > n$ the matrix describing the linear mapping relating the n-dimensional state to the vector

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Guest Editors Torsten Söderström, Paul van den Hof, Bo Wahlberg and Siep Weiland.

* Fax: +43 1 58801 11999.
E-mail address: Dietmar.Bauer@tuwien.ac.at (D. Bauer).

0005-1098/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.11.012

of predictions for the outputs $(y_{t+j})_{j=0,\dots,f-1}$ (conditional on known inputs) is of dimension $fs \times n$ defining a subspace of $\mathbb{R}^{fs}$ via its column space. Also an alternative motivation for the term *subspace methods* exists: A particular class of algorithms tries to estimate the state in the first step. Interpreting least-squares predictions as projections of random variables in an appropriately defined Hilbert space the projections of $(y_{t+j})_{j=0,\dots,f-1}$ given $(u_{t+j})_{j=0,\dots,f-1}$ onto the whole past of inputs and outputs span an n-dimensional subspace providing another interpretation for the origin of the term (for details cf. Picci & Katayama, 1996, and the references therein).

The ways in which this dimensionality reduction to the n-dimensional subspace is exploited differ fundamentally for the proposed algorithms, however, suggesting the consideration of two classes of procedures, see below.[1] This is true even for the class of linear dynamical models analyzed in this paper. In Bauer (2003) for this model class a detailed description of many of the most prominent algorithms proposed in the literature is given. The differences between the various proposed algorithms in many instances are minor details which do not require additional tools for the analysis of the asymptotic properties. In this survey hence it is not attempted to obtain results for all different variants but we will rather restrict attention to the main algorithms described in detail below.

The plan of this paper is as follows: In the next section the stochastic properties of the considered model class are discussed. Section 3 discusses in detail the subspace algorithms that are dealt with in this paper. Subsequently consistency of the various estimators is investigated in Section 4 and questions of asymptotic bias are discussed. Section 5 deals with asymptotic normality and examines expressions for the asymptotic variance under the assumption of correctly specified order of the system. Order estimation techniques are described in Section 6. Topics related to closed-loop operation conditions are briefly discussed in Section 7. Finally Section 8 concludes the paper.

The general style of presentation is intended to be accessible for postgraduate students with some background in statistics. This implies that especially the discussion of the model set is lengthier than otherwise required.

Throughout the paper the following notation will be used: $\|X\|$ denotes the two norm of the matrix X or the Euclidean norm of the vector X, respectively. By $\to$ we denote convergence in probability if not stated explicitely otherwise. A.s. will abbreviate *almost sure*. For a sequence of random matrices F_T the notation $F_T = o(g_T)$ means that $\|F_T\|/g_T \to 0$ a.s., $F_T = O(g_T)$ means that there exists a constant $M < \infty$ such that $\limsup_{T\to\infty} \|F_T\|/g_T \leqslant M$ a.s. $o_P(g_T)$ and $O_P(g_T)$ denote the corresponding in probability versions. X' will denote the transpose of a matrix or a vector and $X > 0$ means that the symmetric matrix X is positive definite. An eigenvalue of maximum modulus of a matrix A will be denoted as $\lambda_{\max}(A)$.

[1] This point of view does not seem to be widely accepted in the community.

2. State space systems

Subspace algorithms have been proposed for a number of different model classes. However, in this paper only the leading case of linear, time invariant, discrete time systems of the form (for $t \in \mathbb{Z}$)

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + K\varepsilon_t, \\ y_t &= Cx_t + Du_t + \varepsilon_t \end{aligned} \tag{1}$$

where the output data $y_t \in \mathbb{R}^s$ and the input data $u_t \in \mathbb{R}^m$ are observed for $t=1,\dots,T$ will be dealt with. Here $(x_t)_{t\in\mathbb{Z}}$ denotes the state sequence and $(\varepsilon_t)_{t\in\mathbb{Z}}$ the unobserved (weak) white noise. The noise ε_t and the input u_s are assumed to be independent for all $t, s \in \mathbb{Z}$, i.e. (with the exception of Section 7) only open-loop operation conditions will be considered. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and $K \in \mathbb{R}^{n\times s}$ are the matrices to be estimated. Additionally the innovation variance $\Omega = \mathbb{E}\varepsilon_t\varepsilon_t'$ needs to be estimated. Throughout the paper the system will be assumed to be stable, i.e. $|\lambda_{\max}(A)| < 1$ will be assumed. Furthermore, the strict minimum-phase condition $|\lambda_{\max}(A-KC)| < 1$ will be imposed.

The following material provides a crash course to linear state space modelling. A more extensive account of the subject can be found in Chapters 1 and 2 of Hannan and Deistler (1988) or in Chapter 4 of Ljung (1999). Subspace methods are inherently using black box modeling, i.e. there is no specific knowledge about the system matrices (A, B, C, D, K) exploited. Let S_n denote the set of all quintuples of system matrices (A, B, C, D, K) where the state is n dimensional and where $|\lambda_{\max}(A)| < 1$ and $|\lambda_{\max}(A - KC)| < 1$ holds. The system equations (1) imply that

$$\begin{aligned} y_t &= Cx_t + Du_t + \varepsilon_t \\ &= C(Ax_{t-1} + Bu_{t-1} + K\varepsilon_{t-1}) + Du_t + \varepsilon_t \\ &= CA^{t-1}x_1 + Du_t + \varepsilon_t + \sum_{j=1}^{t-1} CA^{j-1}[Bu_{t-j} + K\varepsilon_{t-j}] \\ &= CA^{t-1}x_1 + \sum_{j=0}^{t-1} L(j)u_{t-j} + \sum_{j=0}^{t-1} K(j)\varepsilon_{t-j}, \end{aligned} \tag{2}$$

where $L(j) \in \mathbb{R}^{s\times m}$ and $K(j) \in \mathbb{R}^{s\times s}$ are the so-called *impulse response coefficients*. This shows that apart from initial effects (i.e. $CA^{t-1}x_1$) the state space system is only a convenient way to represent the impulse response sequence. Two state space systems $(A, B, C, D, K) \in S_n$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{K}) \in S_{\tilde{n}}$ are called *observationally equivalent* if $D = \tilde{D}$ and $CA^{j-1}[B, K] = \tilde{C}\tilde{A}^{j-1}[\tilde{B}, \tilde{K}]$, $j \in \mathbb{N}$. An equivalent way to describe observational equivalence