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Brief paper

Variance estimation and ranking of target tracking position errors modeled using Gaussian mixture distributions ☆

Lidija Trailović*, Lucy Y. Pao

Department of Electrical and Computer Engineering, University of Colorado, Boulder, CO 80309-0425, USA

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Abstract

In this paper, variance estimation and ranking methods are developed for stochastic processes modeled by Gaussian mixture distributions. It is shown that the variance estimate from a Gaussian mixture distribution has the same properties as the variance estimate from a single Gaussian distribution based on a reduced number of samples. Hence, well-known tools for variance estimation and ranking of single Gaussian distributions can be applied to Gaussian mixture distributions. As an application example, we present optimization of sensor processing order in the sequential multi-target multi-sensor joint probabilistic data association (MSJPDA) algorithm.

Keywords: Variance estimation; Variance ranking; Gaussian mixture; Target tracking; Tracking error

1. Introduction

A popular performance metric used for evaluating tracking algorithms is the root mean square (RMS) position error computed from a variance estimate based on data generated by the tracking algorithm (Frei & Pao, 1998; Pao & Trailović, 2000; Trailović & Pao, 2001, 2004). It has been shown that the position error distribution is much better modeled using *mixtures* of Gaussians rather than a single Gaussian (Trailović, 2002). Estimation and ranking of variances in stochastic processes that generate data with Gaussian distributions are based on well-known properties of the χ^2 and F-distributions (Devore, 2003; Hoel, 1962). The purpose of this paper is to develop the tools for variance estimation and ranking of Gaussian mixture distributions.

The paper is organized as follows. Variance estimation for Gaussian mixture distributions is discussed in Section 2, including derivations of the moment generating function and the probability density function. Section 3 addresses the Gaussian mixture variance ranking. As an application example, we present selection of the optimum order of sensor processing in the sequential multi-sensor joint probabilistic data association (MSJPDA) target tracking algorithm (Bar-Shalom & Fortmann, 1998; Bar-Shalom & Tse, 1975; Frei & Pao, 1998; Pao, 1994; Pao & Trailović, 2000; Trailović, 2002). The results are summarized in Section 4.

2. Variance distribution for Gaussian mixtures

A k-component Gaussian mixture pdf of a random variable x is

$$f_k(x) = \sum_{j=1}^{k} w_j \phi(x;\, m_j, \sigma_j), \tag{1}$$

where $\phi(x;\, m_j, \sigma_j)$ is a Gaussian pdf with mean m_j and standard deviation σ_j and w_j are weights satisfying $\sum_{j=1}^{k} w_j = 1, w_j \geqslant 0$. We consider a class of zero-mean Gaussian mixture distributions with $m_j = 0,\ j = 1, \ldots, k$. Such distributions can be used to model the position error

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* Corresponding author.
E-mail addresses: Lidia.Trailovic@ngc.com (L. Trailović), Lucy.Pao@colorado.edu (Lucy Y. Pao).

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in target tracking algorithms (Trailović, 2002). For the zero-mean Gaussian mixtures, the variance σ_{eq}^2 is

$$\sigma_{eq}^2 = \sum_{j=1}^{k} w_j \sigma_j^2. \tag{2}$$

Given a sample set $\{e_1, \ldots, e_n\}$, the unbiased variance estimator is

$$\hat{\sigma}_{eq}^2 = \frac{1}{n-1} \sum_{i=1}^{n} e_i^2. \tag{3}$$

When samples originate from a single Gaussian distribution, the tools for variance estimation and ranking are based on properties of the χ^2 and the F-distributions (Devore, 2003; Hoel, 1962). Our objective is to develop similar tools for data originating from a k-component zero-mean Gaussian mixture distribution.

Following the derivation for the χ^2 distribution outlined in Hoel (1962), let us define a random variable

$$\varepsilon_n = \sum_{i=1}^{n} x_i^2, \tag{4}$$

where x_i is a zero-mean k-component Gaussian mixture random variable with the pdf in (1). Further, let $g_{kn}(x;\ n)$ be the pdf of the random variable ε_n having a k-component "χ^2-mixture" distribution with n degrees of freedom (i.e., measurements or samples) (Devore, 2003; Hoel, 1962). The scaled random variable

$$\varepsilon = \frac{1}{n-1} \varepsilon_n \tag{5}$$

produces the unbiased variance estimator (3). With the change of variables (5), the objective is to find the density

$$g_k(x;\ n) = (n-1) g_{kn}((n-1)x;\ n). \tag{6}$$

2.1. Moment generating function (mgf)

A pdf is uniquely determined by its moment generating function when it exists (Hoel, 1962). The mgf of the random variable ε_n is defined as

$$M_{\varepsilon_n}(s) = E[\mathrm{e}^{s\varepsilon_n}] = \int_{-\infty}^{\infty} \mathrm{e}^{sx} g_{kn}(x;\ n)\, \mathrm{d}x, \tag{7}$$

where $E[\cdot]$ is the expectation of the random variable. Substituting (4), where x_i are independent random variables with the same zero-mean k-component Gaussian mixture pdf, we have

$$\begin{aligned} M_{\varepsilon_n}(s) &= E[\mathrm{e}^{s\sum_{i=1}^{n} x_i^2}] = \prod_{i=1}^{n} E[\mathrm{e}^{s x_i^2}] = (E[\mathrm{e}^{sx^2}])^n \\ &= (M_{x^2}(s))^n, \end{aligned} \tag{8}$$

$$\begin{aligned} M_{x^2}(s) &= \int_{-\infty}^{\infty} \mathrm{e}^{sx^2} \sum_{j=1}^{k} w_j \phi(x;\ 0, \sigma_j)\, \mathrm{d}x \\ &= \sum_{j=1}^{k} w_j \int_{-\infty}^{\infty} \mathrm{e}^{sx^2} \left(\frac{1}{\sqrt{2\pi}\sigma_j} \mathrm{e}^{-\frac{x^2}{2\sigma_j^2}} \right) \mathrm{d}x \\ &= \sum_{j=1}^{k} w_j (1 - 2\sigma_j^2 s)^{-1/2}. \end{aligned} \tag{9}$$

Combining (8) and (9) yields

$$M_{\varepsilon_n}(s) = \left(\sum_{j=1}^{k} w_j (1 - 2\sigma_j^2 s)^{-1/2} \right)^n = \left(\sum_{j=1}^{k} w_j M_j \right)^n. \tag{10}$$

For $k = 1$, $w_1 = 1$, and $\sigma_1 = \sigma_{eq} = 1$, the mgf (10) reduces to the mgf of the χ^2 distribution (Hoel, 1962),

$$M_{\varepsilon_n}(s)|_{k=1} = M_{\chi^2}(s) = (1 - 2s)^{-n/2}, \tag{11}$$

with the corresponding density

$$g_{1n}(x;\ n) = \frac{x^{(n/2)-1} \mathrm{e}^{-x/2}}{2^{n/2} \Gamma(n/2)}. \tag{12}$$

Unfortunately, for $k > 1$, a closed-form solution for the pdf corresponding to the given mgf (10) is not known.

2.2. χ^2-mixture distribution for large n

Without loss of generality, we assume that the standard deviation (2) of the zero-mean mixture is normalized to $\sigma_{eq} = 1$. By the central limit theorem, the pdf of a sum of n random variables converges to a Gaussian pdf for large n. The mean m and the variance σ^2 can be found from the mgf $M(s)$ as (Hoel, 1962)

$$m = M'(s)|_{s=0} = M'(0), \tag{13}$$

$$\sigma^2 = M''(s)|_{s=0} - m^2 = M''(0) - m^2, \tag{14}$$

where $M'(s)$ and $M''(s)$ are the derivatives of $M(s)$ with respect to s. Applying (13) and (14) to (11), we conclude that the χ^2 pdf with n degrees of freedom converges to the Gaussian pdf

$$g_{1n}(x;\ n) \xrightarrow{n \gg 1} \phi(x;\ m_{\chi_1}, \sigma_{\chi_1}) \tag{15}$$

with the mean $m_{\chi_1} = n$ and the variance $\sigma_{\chi_1}^2 = 2n$. Therefore, by the change of variables (5), the scaled χ^2 pdf converges to the Gaussian pdf

$$g_1(x;\ n) \xrightarrow{n \gg 1} \phi\left(x;\ 1, \sqrt{\frac{2}{n}}\right). \tag{16}$$

Similarly, applying (13) and (14) to (10), we find that the scaled χ^2-mixture density $g_k(x;\ n)$ converges to the Gaussian pdf

$$g_{kn}(x;\ n) \xrightarrow{n \gg 1} \phi(x;\ m_{\chi_k}, \sigma_{\chi_k}) \tag{17}$$