Available online at www.sciencedirect.com
SCIENCE DIRECT®

automatica

Automatica 41 (2005) 1709–1721

www.elsevier.com/locate/automatica

Dynamic programming for constrained optimal control of discrete-time linear hybrid systems☆

Francesco Borrelli[a,*], Mato Baotić[a,1], Alberto Bemporad[b], Manfred Morari[a]

[a] *Automatic Control Laboratory, ETH Zentrum, CH-8092 Zurich, Switzerland*
[b] *Dip. Ingegneria dell'Informazione, University of Siena, 53100 Siena, Italy*

Received 8 July 2003; received in revised form 11 February 2005; accepted 19 April 2005
Available online 21 June 2005

Abstract

In this paper we study the solution to optimal control problems for constrained discrete-time linear hybrid systems based on quadratic or linear performance criteria. The aim of the paper is twofold. First, we give basic theoretical results on the structure of the optimal state-feedback solution and of the value function. Second, we describe how the state-feedback optimal control law can be constructed by combining multiparametric programming and dynamic programming.

Keywords: Hybrid systems; Piecewise affine systems; Optimal control; Dynamic programming; Multiparametric programming

1. Introduction

Recent technological innovations have caused a considerable interest in the study of dynamical processes of a mixed continuous and discrete nature, denoted as hybrid systems. In their most general form hybrid systems are characterized by the interaction of continuous-time models (governed by differential or difference equations), and of logic rules and discrete event systems (described, for example, by temporal logic, finite state machines, if-then-else rules) and discrete components (on/off switches or valves, gears or speed selectors, etc.). Such systems can switch between many operating modes where each mode is governed by its own characteristic dynamical laws. Mode transitions are triggered by variables crossing specific thresholds (state events), by the lapse of certain time periods (time events), or by external inputs (input events) (Antsaklis, 2000). A detailed discussion of different modeling frameworks for hybrid systems that appeared in the literature goes beyond the scope of this paper; the main concepts can be found in Antsaklis (2000), Branicky, Borkar, and Mitter (1998), Bemporad and Morari (1999), Lygeros, Tomlin, and Sastry (1999).

Different methods for the analysis and design of controllers for hybrid systems have emerged over the last few years (Sontag, 1981; Lygeros et al., 1999; Bemporad & Morari, 1999). Among them, the class of optimal controllers is one of the most studied. The approaches differ greatly in the hybrid models adopted, in the formulation of the optimal control problem and in the method used to solve it.

In this paper we focus on discrete-time linear hybrid models. In our hybrid modeling framework we allow (i) the system to be discontinuous, (ii) both states and inputs to assume continuous and discrete values, (iii) events to be both internal, i.e., caused by the state reaching a particular boundary, and exogenous, i.e., forced by a switch to some other operating mode, and (iv) states and inputs to fulfill

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Andrew R. Teel under the direction of Editor H.K. Khalil.

* Corresponding author: Dipartimento di Ingegneria, Università del Sannio in Benevento, Piazza Roma 21, 82100 Benevento, Italy. Tel.: +39 0824 30 5586; fax: +39 0824 32 5246.

E-mail addresses: francesco.borrelli@unisannio.it (F. Borrelli), mato.baotic@fer.hr (M. Baotić), bemporad@dii.unisi.it (A. Bemporad), morari@control.ee.ethz.ch (M. Morari).

[1] Current address: University of Zagreb, 10000 Zagreb, Croatia.

0005-1098/$ - see front matter
doi:10.1016/j.automatica.2005.04.017

linear constraints. We will focus on discrete-time piecewise affine (PWA) models. Discrete-time PWA models can describe a large number of processes, such as discrete-time linear systems with static piecewise-linearities; discrete-time linear systems with discrete states and inputs; switching systems where the dynamic behavior is described by a finite number of discrete-time linear models together with a set of logic rules for switching among these models; approximation of nonlinear discrete-time dynamics, e.g., via multiple linearizations at different operating points.

In discrete-time hybrid systems an event can occur only at instants that are multiples of the sampling time, and many interesting mathematical phenomena occurring in continuous-time hybrid systems such as Zeno behaviors do not exist. However, the solution to optimal control problems is still complex: the solution to the HJB equation can be discontinuous and the number of possible switches grows exponentially with the length of the horizon of the optimal control problem. Nevertheless, we will show that for the class of linear discrete-time hybrid systems we can *characterize* and *compute* the optimal control law exactly *without gridding* the state space.

The solution to optimal control problems for discrete-time hybrid systems was first outlined by Sontag (1981). In his plenary presentation (Mayne, 2001) at the 2001 European Control Conference, Mayne presented an intuitively appealing characterization of the state-feedback solution to optimal control problems for linear hybrid systems with performance criteria based on quadratic and linear norms. The detailed exposition presented in the initial part of this paper follows a similar line of argumentation and shows that the state-feedback solution to the finite time optimal control problem is a time-varying PWA feedback control law, possibly defined over non-convex regions. Moreover, we give insight into the structure of the optimal state-feedback solution and of the value function.

In the second part of the paper we describe how the optimal control law can be efficiently computed by means of multiparametric programming. In particular, we propose a novel algorithm that solves the Hamilton–Jacobi–Bellman equation by using a simple multiparametric solver. In collaboration with different companies and institutes, the results described in this paper have been applied to a wide range of problems (Baotic, Vasak, Morari, & Peric, 2003; Bemporad, Borodani, & Mannelli, 2003; Bemporad, Giorgetti, Kolmanovsky, & Hrovat, 2002; Bemporad & Morari, 1999; Borrelli, Bemporad, Fodor, & Hrovat, 2001; Ferrari-Trecate et al., 2002; Mignone, 2002; Möbus, Baotic, & Morari, 2003; Torrisi & Bemporad, 2004). Simple examples that highlight the main features of the hybrid system approach presented in this paper can be found in Borrelli, Baotic, Bemporad, and Morari (2003).

Before formulating optimal control problems for hybrid systems we will give a short overview on multiparametric programming and on discrete-time linear hybrid systems.

2. Definitions and basic results

We will use the following non-standard definitions:

Definition 1. A polyhedron is a set that equals the intersection of a finite number of closed halfspaces. An open set $\mathscr{R}$ whose closure $\bar{\mathscr{R}}$ is a polyhedron is called open polyhedron. A "neither open nor closed polyhedron" is a neither open nor closed set $\mathscr{R}$ whose closure $\bar{\mathscr{R}}$ is a polyhedron. A non-Euclidean polyhedron is a set whose closure equals the union of a finite number of polyhedra.

Definition 2. A collection of sets $\mathscr{R}_1, \ldots, \mathscr{R}_N$ is a *partition* of a set Θ if (i) $\bigcup_{i=1}^{N} \mathscr{R}_i = \Theta$, (ii) $\mathscr{R}_i \cap \mathscr{R}_j = \emptyset$, $\forall i \neq j$. Moreover $\mathscr{R}_1, \ldots, \mathscr{R}_N$ is a *polyhedral partition* of a polyhedral set Θ if $\mathscr{R}_1, \ldots, \mathscr{R}_N$ is a partition of Θ and the $\bar{\mathscr{R}}_i$'s are polyhedral sets, where $\bar{\mathscr{R}}_i$ denotes the closure of the set $\mathscr{R}_i$.

Definition 3. A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is *PWA* if there exists a partition $\mathscr{R}_1, \ldots, \mathscr{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in \mathscr{R}_i$, $i = 1, \ldots, N$.

Definition 4. A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is *PWA on polyhedra* (*PPWA*) if there exists a polyhedral partition $\mathscr{R}_1, \ldots, \mathscr{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall \theta \in \mathscr{R}_i$, $i = 1, \ldots, N$.

Piecewise quadratic (PWQ) functions and piecewise quadratic functions on polyhedra (PPWQ) are defined analogously.

Definition 5. A function $q : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is a *multiple quadratic function* of multiplicity $d \in \mathbb{N}^+$ if $q(\theta) = \min\{q^1(\theta) \triangleq \theta' Q^1 \theta + l^1\theta + c^1, \ldots, q^d(\theta) \triangleq \theta' Q^d \theta + l^d\theta + c^d\}$, $Q^i > 0$, $\forall i = 1, \ldots, d$ and Θ is a convex polyhedron.

Definition 6. A function $q : \Theta \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is a *multiple PWQ on polyhedra* (multiple PPWQ) if there exists a polyhedral partition $\mathscr{R}_1, \ldots, \mathscr{R}_N$ of Θ and $q(\theta) = \min\{q_i^1(\theta) \triangleq \theta' Q_i^1 \theta + l_i^1\theta + c_i^1, \ldots, q_i^{d_i}(\theta) \triangleq \theta' Q_i^{d_i} \theta + l_i^{d_i}\theta + c_i^{d_i}\}$, $\forall \theta \in \mathscr{R}_i$, $i = 1, \ldots, N$. We define d_i to be the multiplicity of the function q in the polyhedron $\mathscr{R}_i$, and $d = \sum_{i=1}^{N} d_i$ to be the multiplicity of the function q. (Note that Θ is not necessarily convex.)

3. Basics of multiparametric programming

Consider the nonlinear mathematical program dependent on a parameter vector x appearing in the cost function and in the constraints

$$J^*(x) = \inf_z f(z, x) \quad \text{subj. to } g(z, x) \leqslant 0, \quad z \in M, \tag{1}$$