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Conducting cylinders in an external electric field: Polarizability and field enhancement

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ABSTRACT

We obtain a simple and exact expression for the average field E_{ave} in the gap between cylinders of arbitrary radii and separation. For given external field E_0 parallel to the plane of the cylinder axes, E_{ave}/E_0 increases in proportion to $s^{-1/2}$ as the separation s of the cylinders tends to zero. In addition, exact expressions are derived for the longitudinal and transverse polarizabilities of a pair of cylinders, and for their contact values.

1. Introduction

The electrostatics of two conducting *spheres* in an external field is now well-understood, both in the system polarizability [1–6], and in the enhancement of the field in the space between the spheres [5–7].

For a pair of parallel conducting cylinders, there is a substantial literature on *charged* cylinders [8–14]. For *uncharged* cylinders in an external field, the author has recently calculated the longitudinal and transverse polarizabilities of two cylinders of equal radius [15]. Here we consider the general problem of a pair of uncharged cylinders of arbitrary radii and at any separation, in an external field. We shall derive exact expressions for the longitudinal and transverse polarizabilities, and also for the enhancement of the external electric field in the region between the cylinders. We find that the enhancement factor goes to infinity (in the longitudinal case) as the cylinders approach contact.

As in Ref. [15], we shall use bicylindrical coordinates u, v which are related to Cartesian x, y coordinates by the conformal transformation

$$\frac{x+iy}{\ell} = i\cot\frac{v+iu}{2}. \qquad (1)$$

This maps the region exterior to the cylinders onto the rectangle $-u_b < u < u_a$, $-\pi < v < \pi$ in the uv plane. Equating the real and imaginary parts of (1) gives

$$\frac{x}{\ell} = \frac{\sinh u}{\cosh u - \cos v}, \quad \frac{y}{\ell} = \frac{\sin v}{\cosh u - \cos v}. \qquad (2a)$$

Elimination of v from the equation (2) gives circles in the xy plane parameterized by u:

$$(x - \ell\coth u)^2 + y^2 = \frac{\ell^2}{\sinh^2 u}. \qquad (2b)$$

Thus $u = u_a$ corresponds to a circular cylinder parallel to the z-axis, centered on $(\ell\coth u_a, 0)$, with radius $a = \ell/\sinh u_a$. Likewise the circular cylinder $u = -u_b$ $(u_b > 0)$ is centered on $(-\ell\coth u_b, 0)$, and has radius $b = \ell/\sinh u_b$. The scale length ℓ is determined once we specify the distance between the cylinder centers, which we will call c. It is given by

$$\ell = \frac{[(c+a+b)(c-a-b)(c+a-b)(c-a+b)]^{\frac{1}{2}}}{2c}. \qquad (3)$$

The relation (3) follows from

$$c = \ell(\coth u_a + \coth u_b), \quad \sinh u_a = \ell/a, \quad \sinh u_b = \ell/b. \qquad (4)$$

Also from equation (4), u_a and u_b may be explicitly expressed in terms of the cylinder radii a, b and center to center distance c:

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$$\cosh u_a = \frac{c^2 + a^2 - b^2}{2ac}, \quad \cosh u_b = \frac{c^2 - a^2 + b^2}{2bc}. \tag{5}$$

Fig. 1 shows a section of the two cylinders. We shall be considering the effects of an external field along the x or y axes, respectively longitudinal and transverse to the plane containing the cylinder axes. The transverse case is simpler, and is treated first, in Section 2. The longitudinal case is considered in Sections 3 and 4. Section 5 gives the surface charge density on the cylinders, and Section 6 together with the Appendix discuss the polarizabilities of a cylinder pair in close approach and at contact.

2. Cylinder pair in transverse external field

We seek a solution of Laplace's equation $V(x,y)$ which tends to $-E_0 y$ far from the cylinders. The external field $(0, E_0, 0)$ is upwards in Fig. 1, and we specify the potential to be zero at $y = 0$. Both cylinders are thus at zero potential, and the potential is odd in y. Since the relation between the (x,y) and (u,v) coordinates is conformal, any differentiable function of $v + iu$ will satisfy Laplace's equation. Given the symmetries noted above, a possible solution is

$$V(u,v) = -E_0 y + E_0 \ell \sum_{n=1}^{\infty} \sin nv \left\{ A_n e^{nu} + B_n e^{-nu} \right\}. \tag{6}$$

From equation (15) of [15], $y = 2\ell \sum_{n=1}^{\infty} e^{-n|u|} \sin nv$, so

$$V(u,v) = E_0 \ell \sum_{n=1}^{\infty} \sin nv \left\{ A_n e^{nu} + B_n e^{-nu} - 2e^{-n|u|} \right\}. \tag{7}$$

The potential is to be zero on $u = u_a$ and on $u = -u_b$. Since $\sin nv$ form an orthogonal set for integer n, we have

$$A_n e^{2nu_a} + B_n = 2, \quad A_n + B_n e^{2nu_b} = 2, \tag{8}$$

and therefore

$$A_n = 2\frac{e^{2nu_b} - 1}{e^{2nU} - 1}, \quad B_n = 2\frac{e^{2nu_a} - 1}{e^{2nU} - 1}, \quad U = u_a + u_b. \tag{9}$$

The solution with the correct boundary conditions is thus

$$\frac{V}{E_0 \ell} = \frac{-\sin v}{\cosh u - \cos v} + 2 \times \sum_{n=1}^{\infty} \sin nv \left\{ \frac{(e^{2nu_b} - 1)e^{nu} + (e^{2nu_a} - 1)e^{-nu}}{e^{2nU} - 1} \right\}. \tag{10}$$

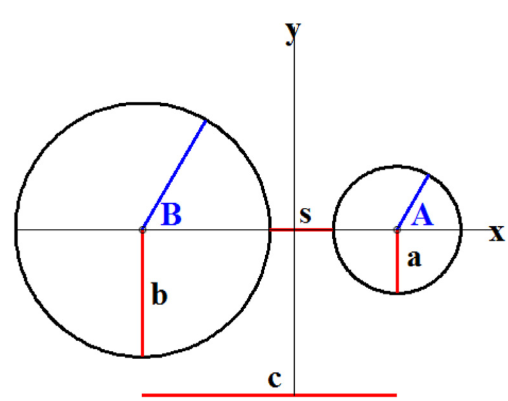

Fig. 1. Two parallel circular cylinders, of radii a and b, specified by $u = u_a$ and $u = -u_b$ in bicylindrical coordinates. The center-to-center distance is c, the smallest separation is $s = c - a - b$. The angles A and B specify azimuthal positions on the two cylinders, as needed for the surface charge densities (Section 5).

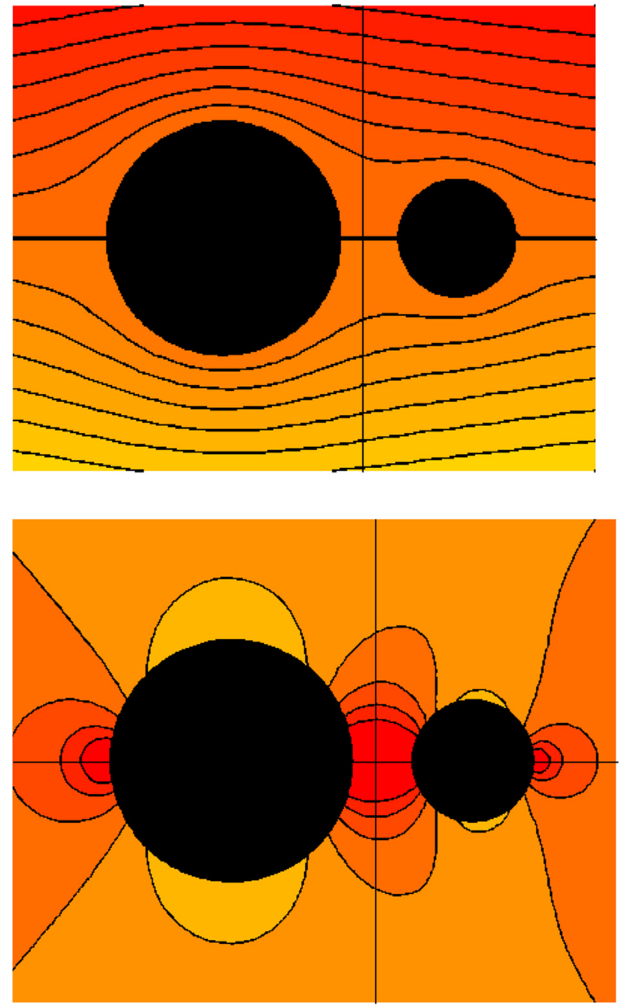

Fig. 2. Cylinders of radii a and $b = 2a$, separated by distance $s = a$ (separation of centers $c = 4a$), in a transverse external field, which is vertical in the figures. For this configuration, the centers of the cylinders are at $(13a/8, 0)$ and $(-19a/8, 0)$. The upper figure shows equipotentials at equal intervals, the lower shows contours of equal intensity $(E/E_0)^2$, increasing in factors of 2 from 1/8 (red) to 4 (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The expression (10) is equivalent to that of Example 83 on p297 of Jeans [16], quoted from a Cambridge Tripos question. Fig. 2 shows the potential distribution around two cylinders in a transverse external field, calculated from equation (10), and also the electric field intensity E^2. The intensity is $E_x^2 + E_y^2 = E_u^2 + E_v^2$, where $E_u \ell = -(\cosh u - \cos v)\partial_u V$ and $E_v \ell = -(\cosh u - \cos v)\partial_v V$, from equations (32) and (35).

We now wish to extract the *transverse polarizability*. (The field enhancement factor is not interesting in the transverse case: the cylinders are at the same potential, and the average field along the shortest line joining them is zero.) To find the transverse polarizability α_T of the cylinder pair, we need to compare the asymptotic form of (10) minus the external field term with that of a dipole $\mathbf{p} = (0, p, 0)$, namely [15] $V_p = 2py/r^2$, where $r = \sqrt{x^2 + y^2}$. From Ref. [15], or directly from equation (2), we see that large r corresponds to both u and v small, with $r \to 2\ell[u^2 + v^2]^{-\frac{1}{2}}$. It then follows that $V_p \to pv/\ell$; this is to be compared with the asymptotic form of the sum in equation (6) as $u, v \to 0$, namely $E_0 \ell v \sum_{n=1}^{\infty} n(A_n + B_n)$. The transverse polarizability $\alpha_T = p/E_0$ is therefore given by

$$\alpha_T = 2\ell^2 \sum_{n=1}^{\infty} n(A_n + B_n) = 2\ell^2 \sum_{n=1}^{\infty} n \frac{e^{2nu_a} + e^{2nu_b} - 2}{e^{2nU} - 1}. \tag{11}$$