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Coulomb potential and energy of a uniformly charged cylindrical shell

Orion Ciftja[a,*], Josep Batle[b]

[a] Department of Physics, Prairie View A&M University, Prairie View, TX 77446, USA
[b] Departament de Física, Universitat de Les Illes Balears, 07122 Palma de Mallorca, Balearic Islands, Spain

A B S T R A C T

Knowledge of electrical potential and energy is important to systems where electrostatic forces play a role. We calculate exactly the electrostatic potential of a uniformly charged cylindrical shell at an arbitrary point in space as well as its electrical energy. The expressions derived are applicable to cylindrical models containing charged particles and certain biological systems with cylindrical symmetry. An explicit analytical formula is provided for the total energy in terms of a class of special functions known as generalized Hypergeometric functions. The expressions when written in one-dimensional integral form are very suitable for numerical calculations.

1. Introduction

One of the most interesting problems in electrostatics is the calculation of the electrostatic potential created by either a finite system of point charges, or by a macroscopic charged body [1–4]. This problem is inherently linked to that of how much electrostatic energy is stored in systems containing interacting charges [5–12]. Systems consisting of point charges interacting with a Coulomb potential are widespread in nature. Therefore, the calculation of the total electrostatic energy, namely, the calculation of the Coulomb self-energy contained in such systems is a problem of utmost importance in many physical disciplines. While the methodology is well known, the possibility to obtain exact analytic results depends heavily into the details of the system under consideration, geometry and how charges are distributed [13,14]. This means that, under general conditions, the problem is very difficult. As a consequence, there are few analytic results that apply to systems with an arbitrary number of point charges. Among few exceptions, we mention the case of one-dimensional (1D) ionic crystals [15].

If charge distribution in a given object is characterized by some charge density, the final answer to the question of what is the electrostatic potential or energy of such a body hinges upon one's ability to calculate the integral expressions for the related quantities. This is the real world of charged bodies made of a variety of different materials that can have any arbitrary shape or form. It turns out that solving the problem of how charge ends up distributed in a conductor or an insulator is analytically impossible if the body under consideration is arbitrarily shaped. In addition to that, finding the equilibrium charge distribution even in a regular body is not a simple problem in general [16]. There are only few cases where the equilibrium charge distribution is known analytically and they comprise the likes of a thin two-dimensional (2D) conducting disk or a three-dimensional (3D) conducting spherical shell. Other scenarios, including the simple-looking problem of what is the exact charge distribution along a 1D wire still do not have a definitive answer [17–20]. A more realistic counterpart to a 1D wire system is either a solid cylinder with a given volume charge distribution or a cylindrical shell, namely, an infinitely thin hollow cylinder containing some surface charge distribution over the lateral surface.

These two systems are very important because many electric devices contain either solid cylinders or cylindrical shells as their components. The equilibrium charge distribution in such systems is impossible to obtain analytically. Therefore, simplifying assumptions are needed. Assuming a uniform charge distribution is often best. Alternatives include assuming a uniform potential (conducting material) and choosing a simple, spatially varying charge distribution (linear, quadratic, etc.). The assumption of uniform charge distribution over the volume of a solid cylinder led to an analytic result for this particular case [21]. In this work, we follow the same assumption and show that exact analytic expressions are also possible for the case of a uniformly charged cylindrical shell. A uniformly charged cylindrical shell model is a natural extension of the uniformly charged ring model with the additional benefit of incorporating a finite length in the third dimension. We introduce mathematical transformations that rely on certain auxiliary functions which enable us to calculate exactly the electrostatic potential and the total energy stored in a uniformly charged cylindrical shell with arbitrary length and radius (and no caps). These results can be useful to numerical studies [22] or studies of finite systems of electrons caged, for instance, in charged nano-tubes or charged nano-cylinders.

The article is organized as follows. In Section 2 we introduce the model and explain the details of a mathematical method that leads to a

* Corresponding author.
E-mail address: ogciftja@pvamu.edu (O. Ciftja).

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closed form expression for the electrostatic potential created by a uniformly charged cylindrical shell at an arbitrary point in space. In Section 3 we expand the earlier mathematical calculations and obtain an exact analytical expression for the total electrostatic energy, the Coulomb self-energy, contained in such a body. In Section 4 we briefly discuss the key findings and present some concluding remarks.

2. Electrostatic potential of a uniformly charged cylindrical shell

We consider a uniformly charged infinitely thin hollow cylinder, namely, a cylindrical shell with radius R and length L. The cylindrical shell under consideration has no caps. The lateral surface of the cylindrical shell is uniformly filled with positive charge, Q. Thus, the surface charge density on the lateral surface of the cylindrical shell is written as:

$$\sigma = \frac{Q}{2\,\pi\,R\,L}\,. \tag{1}$$

Because of the axial symmetry of the problem, we choose a cylindrical system of coordinates where $\vec{r} = \vec{\rho} + \vec{k}\,z$ is a 3D vector, $\vec{\rho} = \vec{i}\,x + \vec{j}\,y$ is its 2D counterpart in polar coordinates and $\vec{i}, \vec{j}, \vec{k}$ are unit vectors for the x, y, z directions, respectively. For such a choice, $x = \rho\cos(\varphi)$, $y = \rho\sin(\varphi)$ where $\rho = |\vec{\rho}\,| \geq 0$ and φ is the polar angle. In a compact notation, we denote the 3D vector as $\vec{r} = (\vec{\rho}, z)$. The coordinative system is chosen in such a way that the cylindrical shell is represented by the following region/domain:

$$\Omega:\ \rho = R\ ;\quad 0 \leq \varphi \leq 2\pi\ ;\quad 0 \leq z \leq L\,. \tag{2}$$

Consider an elementary charge dQ' at the location, $\vec{r}\,' = (\vec{\rho}\,', z')$ somewhere on the lateral surface of the cylindrical shell. The elementary electrostatic potential created by dQ' at some arbitrary point in space represented by vector, $\vec{r} = (\vec{\rho}, z)$ can be written as: $dV(\vec{r}) = k_e\,dQ'/\left|\vec{r} - \vec{r}\,'\right|$ where k_e is Coulomb's electric constant. A schematic presentation of the system under consideration is given in Fig. 1. The electrostatic potential created by the entire cylindrical shell will depend on ρ and z (but not on φ due to axial symmetry) as well as radius R and length L. Therefore, we denote it as $V(\rho, z, R, L)$ and write:

$$V(\rho, z, R, L) = \int_\Omega \frac{k_e\,dQ'}{\left|\vec{r} - \vec{r}\,'\right|} = k_e\sigma R\int_0^L dz'\int_0^{2\pi} d\varphi'\frac{1}{\left|\vec{r} - \vec{r}\,'\right|}\,, \tag{3}$$

where Ω is the domain of integration in Eq. (2) and $\vec{r}\,' = (\vec{\rho}\,', z')$ is such that $\left|\vec{\rho}\,'\right| = R$.

Integrals of this form are often encountered in studies of potential theory, but closed form solutions are rarely possible. In order to calculate the integral above, one expands $1/\left|\vec{r} - \vec{r}\,'\right|$ using the following formula (see pg. 565 of Ref. [23] or pg. 140 of Ref. [24]):

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{m=-\infty}^{+\infty}\int_0^\infty dk\, e^{i\,m\,(\varphi_1 - \varphi_2)}\,J_m(k\,\rho_1)\,J_m(k\,\rho_2)\,e^{-k|z_1 - z_2|}\,, \tag{4}$$

where $J_m(x)$ are Bessel functions of the first kind of integral m-th order, $i = \sqrt{-1}$ is the imaginary number and, as seen from the context, k is a dummy variable (not to be confused with the unit vector, $\vec{k}$ mentioned earlier). Note that, for this specific case, $\vec{r}_1 = \vec{r}$, $\vec{r}_2 = \vec{r}\,'$ and $\rho_2 = \rho' = R$. At this juncture, it is straightforward to notice that integration over the angular variable, φ' can be easily carried out resulting in the expression:

$$V(\rho, z, R, L) = k_e\,\sigma\,(2\,\pi\,R)\int_0^L dz'\int_0^\infty dk\,J_0(k\,\rho)\,J_0(k\,R)\;e^{-k|z - z'|}\,. \tag{5}$$

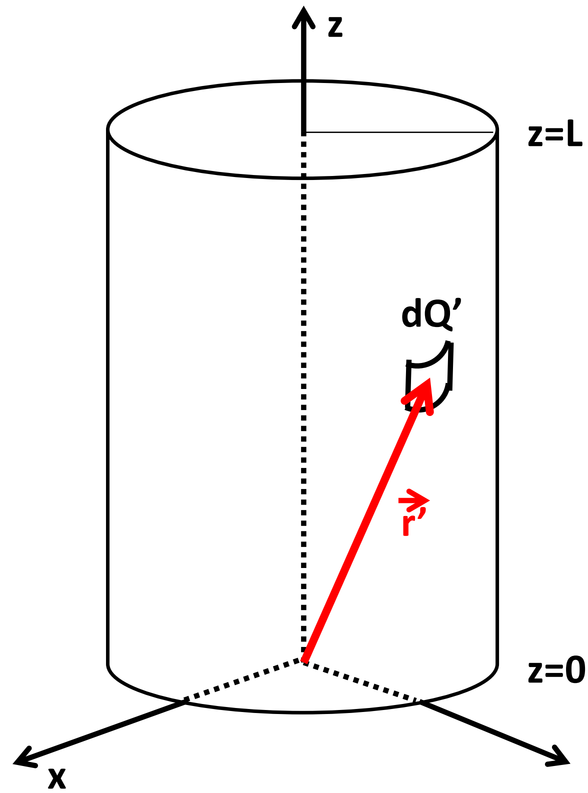

Fig. 1. Schematic view of a uniformly charged cylindrical shell. The elementary charge, dQ' localized at $\vec{r}\,'$ creates an elementary electrostatic potential at some arbitrary point in space.

Note that: $k_e\,\sigma\,(2\,\pi\,R) = k_e\,Q/L$. Therefore, the result in Eq. (5) can be written as:

$$V(\rho, z, R, L) = \frac{k_e\,Q}{L}\int_0^\infty dk\,J_0(k\,\rho)\,J_0(k\,R)\,g(k, z, L)\,, \tag{6}$$

where

$$g(k, z, L) = \int_0^L dz'\,e^{-k|z - z'|}\,, \tag{7}$$

represents an auxiliary function. The 1D integral expression in Eq. (6) is compact and very convenient for numerical calculations. One may also use it as a starting point to derive the corresponding result for a uniformly charged ring in the $L \to 0$ limit. One can verify by L'Hôpital's Rule that:

$$\lim_{L\to 0}\frac{g(k, z, L)}{L} = e^{-k|z|}\,. \tag{8}$$

Based on Eq. (8) one concludes that in the $L \to 0$ limit one has:

$$V(\rho, z, R, L = 0) = k_e\,Q\int_0^\infty dk\,J_0(k\,\rho)\,J_0(k\,R)\,e^{-k|z|}\,. \tag{9}$$

The result in Eq. (9) represents the electrostatic potential created by a uniformly charged ring with radius R and charge Q [see Eq. (10) of Ref. [25]].

From now on, we assume that $L \neq 0$. For this assumption, one has:

$$g(k, z, L) = \frac{1}{k}\left(2 - e^{-k\,z} - e^{+k(z-L)}\right)\ ;\quad L \neq 0\,. \tag{10}$$

Note that $g(k, z = 0, L) = g(k, z = L, L)$. This is a reflection of the fact that $V(\rho, z = 0, R, L) = V(\rho, z = L, R, L)$. The value of the electrostatic potential on the lateral surface of the cylindrical shell can be written as:

$$V(\rho = R, 0 \leq z \leq L, R, L) = \frac{k_e\,Q}{L}\int_0^\infty dk\,[J_0(k\,R)]^2\,g(k, z, L)\,, \tag{11}$$