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On ratio scales

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ABSTRACT

In the past, ratio scales have been a highly controversial issue between physicists and psychologists as regards the need for an empirical addition operation in order to attain them. This aspect is critical since such an operation is often unavailable in psychophysical experiments, as it is in physics, due to its intensive properties. We propose a reconsideration of ratio scales, showing how they can be obtained when ratio and difference empirical relations are available and when they satisfy proper compatibility conditions. We call such structures "intensive" and provide deterministic and probabilistic axiomatizations in the finite case. We also address the practical applications of these ideas to perceptual measurements.

1. Background and state of the art

In the past, ratio scales have been a highly controversial issue between physicists and psychologists [23].[1]

Why are they are so important?

Campbell [1] notes that they enable measurement with a minimum degree of arbitrariness since once a conventional unit has been chosen, the scale is entirely fixed. Stevens [3] points out that statements concerning ratios, or percentages, are meaningful for them, whilst they are not for weaker scales. Finkelstein [17] says that they are apt to represent rich relational structures, conformal to what he calls the paradigm of "strongly defined quantities".

Why have they been controversial?

The main question is whether is it possible to obtain a ratio scale when an *empirical-addition* operation is not available for the property of interest. "I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of *addition* as applied to sensation" – Guild notes in the final Report of the Committee appointed by the British Association for the Advancement of Science, for discussing and reporting about the "quantitative estimation of sensory events" [2]. In fact, whilst addition is often possible in physics – it is possible to add lengths, masses, electrical resistances – this is not usually the case for perceptual properties. Even in physics it is possible to distinguish between *extensive* and *intensive* quantities [5]. The former are closely related to the space–time extension of bodies while the latter are not. For example, the mass of an homogeneous body is proportional to its spatial extension (its volume), whilst its density is independent of it. Cunietti probes this subject further [14] by recalling Kant's vision [13]. "By an extensive quantity, I mean one such that the representation of its parts makes possible the representation of the whole" – writes the philosopher in his "Critique of pure reason" – whilst "Now I call an *intensive quantity* any quantity which is apprehended only as a unity, and the quantifiability of which can be represented only as its distance from negation = 0". In other words, an extensive quantity may be thought as a sum of parts whilst an intensive one as the grade of a sensation. (A single realization of) an extensive quantity may be geometrically represented by a segment, S; an intensive one by a point, P, on an oriented semi straight-line, as shown in Fig. 1. The associated geometrical features are

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[1] References are listed in chronological order (of first publication) in order to provide an overview of the historical development of the subject.

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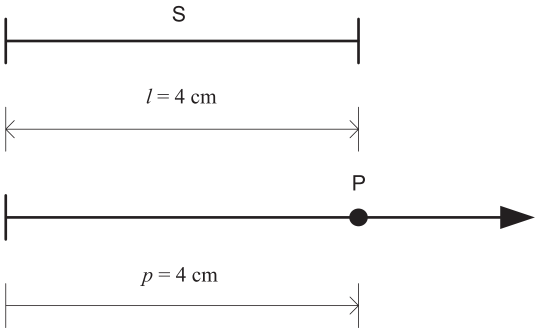

Fig. 1. An extensive quantity may be represented by a segment, *S*, an intensive one by a point, *P*. Following Kant's line of thought, we may note that the segment is perceivable since it has a spatial extension, whilst a point is not (to make the point visible it had to be magnified!). The point represents the intensity, or grade, of the sensation associated with a perception and is separated by the origin of the reference axis (corresponding to a null sensation) by a continuum of grades (*quantitas qualitatis est gradus*).

length, *l*, and position, *p*: they are both expressed in meters, but their meaning is significantly different.

So the problem now is as follows: how can an intensive quantity be measured on a ratio scale? The classical answer, provided by Campbell [1] is based on the distinction between *fundamental* and *derived* quantities. Fundamental quantities can be directly measured thanks to their internal properties, in particular, thanks to the addition operation that allows a reference scale to be built, whose elements are multiples or submultiples of one unitary element. Once the scale is available, measurement can be performed by comparing unknown objects with the reference scale. Derived quantities instead can be measured indirectly using various natural laws linking them to other measurable quantities. A scientific discipline, such as mechanics, electromagnetism or overall physics, is based on a system of measurable quantities, some of which are fundamentals, the others derived. Note that at least one fundamental quantity is required for the system to be consistent and self-contained. In this traditional approach, non-additive intensive quantities can *only be indirectly* measured as derived quantities.

Subsequently, Stevens made *two key contributions* to this subject (a genius he was, undoubtedly!). Firstly, he introduced a method, known as *magnitude estimation* [4], for directly measuring the intensity of a sensation. Secondly, in his famous theory of measurement scales [3], he indicated "equality of differences" together with "equality of ratios" as distinctive empirical properties allowing measurement on a ratio scale. Unfortunately, he did not provide any axiomatization in support of this last claim. These two contributions generated two distinct lines of research.

Magnitude estimation has been widely studied thenceforth, both experimentally and theoretically. Axiomatizations have been attempted [9], including a conspicuous contribution by Narens [15]. His approach has been checked experimentally [16], generalized [18] and checked again in view of such a generalization [19]. Basically, in this line of research, conditions are investigated for persons to act as measuring instruments when performing magnitude estimation [25].

The second line of research concerns Stevens's claim that the empirical assessment of *both* differences and ratio can yield a ratio scale when *physical addition cannot be assumed*, and this is even more relevant to our purpose. These studies have led to the axiomatization of the so-called ratio/difference representation [7,12] which has been theoretically and experimentally [6,8,10,11] studied to a certain extent.

We will proceed in this latter direction by first developing a deterministic theory for finite intensive structures, where we show how ratio scales can be attained by the *internal*properties of such structures, that *do not include addition*. Then we provide a probabilistic version of this theory in order to properly account for measurement uncertainty. Although our goal here is mainly theoretical, we nonetheless present, in the last part of the paper, a new measurement method that we have called robust magnitude estimation, and discuss its implementation by reporting on experiments that have been performed in our Measurement Laboratory.

2. Beyond additivity

The usual way of obtaining a ratio scale is through an empirical *extensive structure*, that may be formally defined as a triple $(A, \succsim, \circ)$, where *A* is a set of "objects" (events, persons) manifesting some property, *x*, under consideration, $\succsim$ is a weak order relation among objects and $\circ$ is an (empirical) addition operation. A representation theorem for such a structure reads

$$a \sim b \circ c \Longleftrightarrow m(a) = m(b) + m(c), \tag{1}$$

that is, element *a* is equivalent to the empirical sum of *b* and *c*, if and only if the measure of *a* equals the sum of the measures of *b* and *c*. The associated scale is *ratio*, since the measure function *m* may safely undergo any similarity transformation

$$m' = \alpha m, \tag{2}$$

where $\alpha > 0$, which basically consists in a change of the unit of measurement [7,9]. Note that in such a structure there is no "native", so to speak, ratio relation. Rather, ratio is inferred by the measures and such inference is "meaningful" since the scale is ratio. In other words, we say that the mass of *a* is twice the mass of *b*, if $m(a) = 2m(b)$.

A good question now is: *is it possible to obtain a ratio scale through the internal properties (empirical intra-relations) of the characteristic under examination, when they do not include empirical addition?*[2] Indeed this is possible and the empirical *structures*that exhibit such properties are known as *intensive*.

[2] Note that we use the term property with two meanings throughout this paper: either to denote the characteristic we want to measure, which in the current edition of the International Vocabulary of Metrology is called a property, or its characterizing empirical relations. This is the price to be paid for using such a standardized vocabulary: elsewhere we have used the term "characteristic", for property in the first sense, in order to avoid this ambiguity [23].