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The role of fuzzy scales in measurement theory

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ABSTRACT

The introduction of the representational theory of measurement by Stevens initiated a new way to understand what measurement is and was followed by an intense scientific activity. Ludwik Finkelstein mainly contributed to this activity through several synthetic surveys and his formalisation of this theory includes a generalisation of the representation of measurement values to non-numerical sets. The role of group theory in the measurement theory was suggested by Stevens in his seminal paper. Such a role was explored by Narens and Luce for the ordered scales. The studied groups are homomorph to groups acting on real numbers, and other possible scales remain unexplored. For example, the metrical scales, introduced by Coombs, are built on distances and do not fit the classic classification of scale. Initially devoted to psychophysical measurement, metrical scales now appear in various fields, such as colour measurement or software measurement and need to be studied in more detail. The purpose of this paper is to revisit the group-based classification of scales and to show how such a classification includes metrical scales and more specifically fuzzy scales.

1. Preface

We met Ludwik Finkelstein for the first time twenty years ago. During my Phd examination, Professor Finkelstein, as a jury member, asked me how his studies on measurement scales could be improved to help me. Ten years later, I finally understood his encouragement and decided to re-open this field with Laurent. This paper, as an instantaneous state of our studies and inspired by the enthusiasm and kindness of Ludwik Finkelstein, is dedicated to his memory.

2. Introduction: The representational theory of measurement

Scales were introduced to model the link between physical quantities and information entities created by the measurement process. The representational theory of measurement was proposed by Stevens in 1946 as a classification of scales based on their mathematical properties [1]. After controversial approaches, this classification is now commonly accepted as the most significant for scale-type analysis. In 1975, Ludwik Finkelstein proposed a general formal approach where the limitation of the representation of measurement results by numbers is worked around by a generalisation to a symbolic representation [2].

2.1. Introduction of the theory

In his seminal paper, Stevens proposed to classify scale into four types: nominal, ordinal, interval and ratio scales (see Table 1). This classification is driven by the definition, for each scale type, of a set of relations on representations. Stevens also associated a group structure to each scale type. The set of mathematical transformations which leave the scale-form to be invariant has a group structure that characterises the scale type. Such a mathematical transformation modifies the measurement result given by a scale into another measurement result.

A basic empirical operation is a relation on physical quantities that has a representation in the representational space. An admissible transformation is a function on scales

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Table 1
First proposal for scale classification.

Scale	Basic empirical operation	Mathematical group structure	Admissible transformation
Nominal	Determination of equality	Permutation group	$y = f(x)$, f is a bijection
Ordinal	And determination of greater or less	Isotonic group	$y = f(x)$, f is a monotonic increasing function
Interval	And determination of equality of intervals or differences	General linear group	$y = ax + b$, $a > 0$
Ratio	And determination of equality of ratios	Similarity group	$y = ax$, $a > 0$

that preserves the link between the basic empirical relations and their representation. Such a transformation is defined on the set of measurement results. The mathematical group structure of a scale is defined by the group of admissible transformations. These four scale types are ordered by the «sub-group of» relation between groups.

2.2. Formalisation of the theory

The first contribution of Ludwik Finkelstein to the representational theory of measurement was to propose a formal approach of the definition of measurement. He suggested generalising the universe of discourse of measurement values from sets of numerals to sets of symbols. He then defined all scales by a symbolism expressed as:

$$C = \langle X, S, M, R_X, R_S, F \rangle, \tag{1}$$

where

- X refers to a set of quantity manifestations and R_X is a set of relations on X.
- S refers to a set of information entities, and R_S is a set of relations on S.
- M, called the representation, is a mapping from X to S.
- F is an injective mapping with domain R_X and range R_S.

The relational structures $\langle X,R_X\rangle$ and $\langle S,R_S\rangle$ respectively denote an empirical relational system and a representational relational system. The set R_X is the set of relations that are supposed to exist on the empirical set of quantity manifestations. In fact this knowledge comes from the theory chosen to abstract the quantity. We are reminded that a theory is made of entities and affirmations, i.e. axioms or theorems, linking these entities [3]. The set R_X is then part of the theory according to the fact that the set R_S is chosen such that F is a bijection. The mapping M is then a *homomorphic mapping* or *homomorphism* in the sense that it preserves the relational structure. This constraint is expressed by the representation theorem:

$$\forall x_1, \ldots, x_n \in X, F(r_X) = r_S,$$
$$r_X(x_1, \ldots, x_n) \Longleftrightarrow r_s(M(x_1), \ldots, M(x_n)) \tag{2}$$

where r_X is a relation of empirical relational system $\langle X,R_X\rangle$, and r_S is a relation of representational relational system $\langle S,R_S\rangle$: $r_X \in R_X$, $r_s \in R_S$.

The term *homomorphism* must be interpreted in its wider sense, that is as a morphism that preserves a set of relations, and not as a synonym of *group homomorphism*.

The mapping M respecting (2) is not unique, and any application f such that $foM = f(M)$ respects (2) is an *admissible transformation*.

$$\forall x_1, \ldots, x_n \in X, F(r_X) = r_S,$$
$$r_X(x_1, \ldots, x_n) \Longleftrightarrow r_s(f(M(x_1)), \ldots, f(M(x_n))) \tag{3}$$

The scales stay ordered within four types, each type being associated to a class of admissible transformations.

As mentioned by Finkelstein the concept of scale is a bridge between reality, i.e. the empirical world, and our abstract representation of this reality driven by a theory [4]. For any kind of measurement, that is strongly, weakly or widely defined [5], a theory is needed, even if it is a poor theory, i.e. with only few theorems. Actually, a scale is defined by the theory chosen for the abstract world that may represent the empirical world. This theory fixes the class of admissible transformations, and the validity of the scale is directly linked to the validity of the theory.

2.3. The role of groups in scale classification

The representational theory of measurement has inspired several studies, especially for ordinal scales that represent a central interest in psychophysics (see [6,7] for surveys). The link between scale types and group structures remained unexplored until 1987, when Luce and Narens studied measurement scales on continuous spaces [8]. More recently, Narens showed the importance of an approach based on group theory to analyse the representational theory of measurement [9]. This tendency confirms the need to take into account, at a higher level of abstraction, the incidence of the theory on the property of the measurement scale.

The representation theorem induces a class of admissible transformations. Admissible transformations f are in fact automorphisms that preserve the relational structure $\langle S, R_S\rangle$. The classification of scale types is clearly defined with the classification of admissible transformations. As admissible transformations are automorphisms, their class can be studied within the context of the group theory.

In the case of nominal scales, the representational relational system is a relational system $\langle S, = \rangle$ and the set $\mathscr{F}$ of admissible transformations defines the group $(\mathscr{F}, \text{o})$ on S where o is the composition of functions. This group is the symmetric group $Sym(S)$, i.e. the group of all permutations on S which is also the group of all bijections on S. Indeed, by definition, any bijection on S preserves the equality relation:

$$\forall a, b \in S, a = b \Longleftrightarrow f(a) = f(b) \tag{4}$$

The set of admissible transformations of any other scale type defines a subgroup of this general group.

Let $\mathscr{F}$ be a set of admissible transformations preserving the equality on S and r another relation on S. As seen before, the preservation of the equality implies that elements of the set $\mathscr{F}$ are bijections. The closure of $\mathscr{F}$ by the composition operator o is verified as follows: