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Ordinal measurement, preference aggregation and interlaboratory comparisons

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ABSTRACT

The classical problem of a single consensus ranking determination for m rankings of n alternatives has a potential of wide applications in information technologies, and particularly in measurement and instrumentation. The Kemeny rule is one of deeply justified ways to solve the problem allowing to find such a linear order (Kemeny ranking) of alternatives that a distance (defined in terms of a number of pair-wise disagreements between rankings) from it to the initial rankings is minimal. But the approach can result in considerably more than one optimal solutions what can reduce its applicability. By computational experiments outcomes, the paper demonstrates that a set of Kemeny rankings cardinality can be extremely large in small size cases ($m = 4, n = 15 \ldots 20$) and, consequently, special efforts to build an appropriate convoluting solution are needed. Application of the model to one of practical metrological problems, such as interlaboratory comparisons, is proposed and examined.

1. Introduction

In a series of earlier papers [1–6] by the author it was shown that a consideration of an ordinal scale measurement should involve notations of preference (particularly, in form of ranking or weak order) and consensus binary relations. In doing so, a measurement result on the ordinal scale should be the entire ranking of n objects and the ranking is one of elements of the weak order space. From the Representational Measurement Theory point of view, the preference aggregation problem could be seen as a particular case of the general conjoint measurement problem, see, for example [7,8]. This way of thinking is in accordance to the definition by Finkelstein [9–11]: "Measurement is, in the wide sense, an objective, empirical process of establishing a correspondence between properties of objects and events of the real world and a set of symbols and relations. The correspondence is such, that when a symbol is assigned to a manifestation of the property and another symbol is assigned to another manifestation of the same property, then the relation between the two symbols corresponds to a relation between the two manifestations of the property".

A single consensus ranking determination for m rankings (voters), possibly including ties, of n alternatives (candidates) is a classical problem that has been intensively investigated firstly as a Voting Problem in the framework of Social Choice Theory since the late XVIII century.

Condorcet in 1785, see [12], proposed a very natural rule for the consensus ranking determination: if some alternative obtains a majority of votes in pair-wise contests against every other alternative, the alternative is chosen as the winner in the consensus ranking. The Condorcet approach is widely recognized as the best rule for the consensus ranking determination, however, the binary relation defined by the *Condorcet rule* is not necessarily transitive, i.e. it can be for some consensus ranking β that $a_i \succ a_j$ and $a_j \succ a_k$ while $a_k \succ a_i$; $a_i, a_j, a_k \in \beta$. This *Condorcet paradox* may occur rather frequently, for example its chances are higher than 50% at $3 \leqslant m \leqslant \infty$ and $2 \leqslant n \leqslant 10$, if m is even; presence of ties reduces the probability, see, e.g. [13].

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The *Kemeny rule* [14] is considered to be a reasonable way to get over the difficulty as it allows to find such a linear order (Kemeny ranking) β of alternatives that a distance (defined in terms of a number of pair-wise disagreements between rankings) from β to the initial rankings is minimal. But, in turn, the approach has two drawbacks:

- The Kemeny Ranking Problem (KRP) had been proven to be *NP*-hard [12,15–17].
- It may have considerably more than one optimal solutions.

The former is not so disturbing since, for reasonable problem sizes (up to $n < 30\ldots50$), there are exact algorithms for them to be effectively applied, see, for example [6,12,16–19]. Strangely enough, the latter blemish has been given short shrift by researchers despite its importance for the problem applicability. In fact, multiple optimal solutions may rank the alternatives in significantly different ways what can absolutely destroy a positive effect of a potential problem application.

Currently, the model has numerous interpretations and (or a potential for) applications in different domains, such as information retrieval, collaborative filtering [15], multiagent choice and multisensor fusion [15,20], hemometrics [21], digital image processing and pattern recognition [22,23], quality assessment and management [24], sport competitions judging [16,17], multiple criteria (or group) decision making [25], etc. However, the model being singularly fertile of deep measurement theoretical ideas does not have applications in real metrological practice.

The aim of this paper is first to demonstrate that a set of Kemeny rankings cardinality can be extremely large even in cases where $m = 4$ and $n = 15\ldots20$, and, consequently, special efforts to build some appropriate *convoluting solution* are needed. Second, it will be shown how the KRP-based model could be potentially applied to the interlaboratory comparisons problem. Interlaboratory comparisons need a reference value of the measurand to be assigned. It is necessary to have some procedure that allows to determine the reference value at a maximum number of participating laboratories results to be included into the determination and, at the same time, unreliable laboratory results must be disregarded.

The paper is organized as follows. In Section 2, after the KRP statement, an exact algorithm to find all Kemeny rankings for the given preference profile is briefly described. Some intriguing outcomes of computational experiments supporting the declared paper objective are reported and discussed. It is shown in Section 3 that a procedure of interlaboratory comparison can be implemented using the preference aggregation approach. Section 3 also provides a probabilistic way to justify the value m of the comparison participating laboratories.

2. Kemeny ranking problem formulation and solution

This section results were first published in the conference paper [26]. In this section we will use the following symbols:

A	$\{a_1,a_2,\ldots,a_n\}$: a set of n alternatives
Λ	$\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$: a set of m rankings (preference profile)
R	$[r_{ij}]$: an $(n \times n)$ ranking matrix
P	$[p_{ij}]$: an $(n \times n)$ profile matrix
$d(\lambda_k,\lambda_l)$	a distance between two rankings λ_k and λ_l
$D(\lambda,\Lambda)$	a distance between arbitrary ranking λ and profile Λ (Kemeny distance)
Π	a set of all $n!$ linear (strict) order relations $\succ$ on A
β	Kemeny ranking (consensus relation), $\beta \in \Pi$
B	$\{\beta_1,\beta_2,\ldots,\beta_{N_{\text{kem}}}\}$: a set of Kemeny rankings, $B \subset \Pi$
N_{kem}	number of Kemeny rankings for the given profile Λ
D_{least}	a least distance from Λ to some linear order
$\mathbf{N_n}$	$\{1,2,\ldots,n\}$: first n natural numbers
S	$\{s_1,s_2,\ldots,s_K\}$: a partial solution (leader) of the KRP
K	$0,\ldots,n-1$: a level of a search tree
N_{nds}	a total number of the search tree nodes generated
T	$\{t_1,t_2,\ldots,t_{K-n}\} = \mathbf{N_n} \setminus S$: a complement of S
D_{low}	an estimate of Kemeny distance for the ranking with leader S (lower bound)
D_{u}	a minimal value of Kemeny distance for generated to the moment complete solutions (upper bound)

2.1. Problem statement

Suppose we have m *rankings* provided by m experts (voters, focus groups, criteria, etc.) on set A of n *alternatives* (candidates). Then the preference profile Λ consists of m rankings (*weak orders*) $\lambda = \{a_1 \succ a_2 \succ \ldots \sim a_s \sim a_t \succ \ldots \sim a_n\}$, each may include a *strict preference* relation $\succ$ and an *indifference* relation (or *tie*) $\sim$.

The ranking λ can be represented by the *ranking matrix* R, rows and columns of which are labeled by the alternatives' numbers and $r_{ij} = 1$ if $a_i \succ a_j$; $r_{ij} = 0$ if $a_i \sim a_j$; $r_{ij} = -1$ if $a_i \prec a_j$. Then the symmetric difference distance function [14] between two rankings λ_k and λ_l is defined by formula

$$d(\lambda_k,\lambda_l) = \sum_{i<j} |r_{ij}^k - r_{ij}^l|, \tag{1}$$

where only elements of the upper triangle submatrix, r_{ij}, $i < j$, are summed up.

Then a distance between arbitrary ranking λ and profile Λ can be defined as follows:

$$D(\lambda,\Lambda) = \sum_{k=1}^{m} d(\lambda,\lambda_k) = \sum_{i<j}\sum_{k=1}^{m} |r_{ij}^k - r_{ij}| = \sum_{i<j}\sum_{k=1}^{m} d_{ij}^k, \tag{2}$$

Supposing $r_{ij} = 1$ for all $i < j$ that corresponds to the natural linear order $a_1 \succ a_2 \succ \ldots \succ a_n$, it is clear that for any $k = 1,\ldots,m$ we have $d_{ij}^k = |1-1| = 0$ if $a_i^k \succ a_j^k$; $d_{ij}^k = |0-1| = 1$ if $a_i^k \sim a_j^k$ and $d_{ij}^k = |-1-1| = 2$ if $a_i^k \prec a_j^k$.

Then the *profile matrix* P can be defined where