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Phase estimation using a state-space approach based method

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ABSTRACT

The paper demonstrates a phase estimation method in fringe analysis. The proposed method relies on local polynomial phase approximation and subsequent state-space formulation. The polynomial approximation of phase transforms phase extraction into a parameter estimation problem, and the state-space modeling allows the application of Kalman filter to estimate these parameters. The performance of the proposed method is demonstrated using simulation and experimental results.

1. Introduction

For the application of interferometric techniques in optical metrology, fringe analysis [1] plays a crucial role. The major aim in fringe analysis is the reliable extraction of phase, since the information about the measured physical quantity is usually encoded in the phase of a fringe pattern. One of the prominent techniques for phase estimation is the phase-shifting approach [2,3], where multiple interferograms or frames with successive phase incrementation are captured. However, the requirement of multiple frames is an important limitation of this approach, and could render the practical implementation difficult. Accordingly, for phase estimation using a single frame, several spatial fringe analysis methods based on the Fourier transform [4], wavelet transform [5,6], dilating Gabor transform [7], recurrence algorithm [8,9], regularized phase tracking [10], windowed Fourier transform [11], Hilbert transform [12,13], high-order ambiguity function [14], subspace method [15], phase differencing operator [16], etc., have been proposed.

In this paper, we propose an elegant spatial fringe analysis method for phase estimation. The method is presented in the context of digital holographic interferometry (DHI), which is a prominent optical technique for analyzing the deformation of a diffuse object. The theory of the proposed method is outlined in the next section. Simulation and experimental results are presented in Section 3. Finally, conclusions are presented in Section 4, followed by acknowledgments.

2. Theory

In DHI, two holograms are digitally recorded, corresponding to object states before and after deformation. The complex amplitudes for the object wave-fields are obtained using numerical reconstruction through discrete Fresnel transform [17]. Subsequently, the interference field is obtained by multiplying the post-deformation complex amplitude with the conjugate of the pre-deformation complex amplitude. The interference field in DHI can be expressed as

$$\Gamma(x,y) = a(x,y)\exp[j\phi(x,y)] + \eta(x,y) \tag{1}$$

where $a(x,y)$ is the amplitude, $\phi(x,y)$ is the phase and $\eta(x,y)$ is the additive white Gaussian noise. For a given column x, we have

$$\Gamma(y) = a(y)\exp[j\phi(y)] + \eta(y) \tag{2}$$

In the proposed method, phase is modeled as a local polynomial. In other words, $\Gamma(y)$ is divided into say N_w segments, and in each segment, the phase is approximated as a polynomial of degree M. Hence, for a segment i, we have

$$\Gamma_i(y) = a_i(y)\exp[j\phi_i(y)] + \eta_i(y) \quad \forall y \in [1, N_s] \tag{3}$$

$$\phi_i(y) = \sum_{m=0}^{M} \alpha_m y^m \tag{4}$$

where N_s is the size of the segment. From above equation, it is clear that the parameters required to retrieve phase are the polynomial coefficients $(\alpha_0, \ldots, \alpha_M)$.

To estimate these coefficients, a state-space model [18] is applied. Accordingly, using a Taylor series expansion of order M, we have

$$\phi_i(y+1) = \phi_i(y) + \frac{1}{1!}\phi_i^{(1)}(y) + \cdots + \frac{1}{M!}\phi_i^{(M)}(y)$$

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$$= \sum_{p=0}^{M} \frac{1}{p!} \phi_i^{(p)}(y) \tag{5}$$

where the superscript '(p)' denotes the derivative of order p. From above equation, we have

$$\phi_i^{(r)}(y+1) = \sum_{p=r}^{M} \frac{1}{(p-r)!} \phi_i^{(p)}(y) \tag{6}$$

Subsequently, the state vector is chosen as

$$\mathbf{x}(y) = [a_i(y)\ \phi_i(y)\ \phi_i^{(1)}(y)\ \cdots\ \phi_i^{(M)}(y)]^T \tag{7}$$

where 'T' indicates the vector transpose. Clearly, $\mathbf{x}(y)$ is a $(M+2)\times 1$ vector with elements $\{x_1,x_2,\ldots,x_{M+2}\} = \{a_i(y),\phi_i(y),\ldots,\phi_i^{(M)}(y)\}$. The polynomial coefficients are related to the state vector as

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{M+2} \end{bmatrix} = \mathbf{U}(y) \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{bmatrix} \tag{8}$$

or equivalently

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{M+2} \end{bmatrix} = \begin{bmatrix} 1 & y & y^2 & \cdots & y^M \\ 0 & 1 & 2y & \cdots & My^{M-1} \\ 0 & 0 & 2 & \cdots & M(M-1)y^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M! \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{bmatrix} \tag{9}$$

From above equations, it is clear that the matrix $\mathbf{U}(y)$ shows how the phase and its derivatives up to order M are connected to the coefficients.

The state-space analysis is characterized by two sets of equations: (1) *state transition equation*, which describes the evolution of the state vector for successive states and (2) *observation equation*, which exhibits the relationship between the state vector and measurement. Using Eqs. (5) and (6), the state transition equation can be expressed as

$$\mathbf{x}(y+1) = \mathbf{F}\mathbf{x}(y) \tag{10}$$

where

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1/1! & \cdots & 1/M! \\ 0 & 0 & 1 & \cdots & 1/(M-1)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{11}$$

For the observation equation, the measured quantity is the complex signal $\Gamma_i(y)$. Accordingly, using Eq. (3), the observation equation is given as

$$\mathbf{z}(y) = \mathbf{h}(\mathbf{x}(y)) + \mathbf{w}(y) \tag{12}$$

where

$$\mathbf{z}(y) = \begin{bmatrix} \mathrm{Re}\{\Gamma_i(y)\} \\ \mathrm{Im}\{\Gamma_i(y)\} \end{bmatrix} \tag{13}$$

$$\mathbf{h}(\mathbf{x}(y)) = \begin{bmatrix} a_i(y)\cos(\phi_i(y)) \\ a_i(y)\sin(\phi_i(y)) \end{bmatrix} = \begin{bmatrix} x_1\cos(x_2) \\ x_1\sin(x_2) \end{bmatrix} \tag{14}$$

$$\mathbf{w}(y) = \begin{bmatrix} \mathrm{Re}\{\eta_i(y)\} \\ \mathrm{Im}\{\eta_i(y)\} \end{bmatrix} \tag{15}$$

with 'Re' and 'Im' denoting the real and imaginary parts.

Given the state transition and observation equations as in Eqs. (10) and (12), the Kalman filter [19] is used for state estimation. Accordingly, the estimates of the state vector $\mathbf{x}(y)$ and the corresponding error covariance matrices, before and after considering the measurement $\mathbf{z}(y)$ are denoted as

$$\hat{\mathbf{x}}^-(y) = a\ priori\ \text{estimate} \tag{16}$$

$$\hat{\mathbf{P}}^-(y) = E[(\mathbf{x}-\hat{\mathbf{x}}^-)(\mathbf{x}-\hat{\mathbf{x}}^-)^T] \tag{17}$$

$$\hat{\mathbf{x}}^+(y) = a\ posteriori\ \text{estimate} \tag{18}$$

$$\hat{\mathbf{P}}^+(y) = E[(\mathbf{x}-\hat{\mathbf{x}}^+)(\mathbf{x}-\hat{\mathbf{x}}^+)^T] \tag{19}$$

with 'E' indicating the expectation operation. Since the observation equation is nonlinear, the extended Kalman filter (EKF) algorithm [20] is applied, as shown below:

1. The nonlinear function $\mathbf{h}(\mathbf{x}(y))$ in Eq. (14) is linearized around the a priori state estimate using a first order Taylor series

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(\hat{\mathbf{x}}^-) + \mathbf{J}_h(\hat{\mathbf{x}}^-)(\mathbf{x}-\hat{\mathbf{x}}^-) \tag{20}$$

where $\mathbf{J}_h$ is the Jacobian matrix given as

$$\mathbf{J}_h(\hat{\mathbf{x}}^-) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\hat{\mathbf{x}}^-} = \begin{bmatrix} \cos(\hat{x}_2^-) & -\hat{x}_1^-\sin(\hat{x}_2^-) & 0 & \cdots & 0 \\ \sin(\hat{x}_2^-) & \hat{x}_1^-\cos(\hat{x}_2^-) & 0 & \cdots & 0 \end{bmatrix} \tag{21}$$

2. The Kalman gain $\mathbf{K}(y)$ is computed as

$$\mathbf{K} = \hat{\mathbf{P}}^-\mathbf{J}_h^T(\mathbf{J}_h\hat{\mathbf{P}}^-\mathbf{J}_h^T + \mathbf{R})^{-1} \tag{22}$$

with $\mathbf{R}$ being the noise covariance.

3. The a posteriori estimates are obtained as

$$\hat{\mathbf{x}}^+(y) = \hat{\mathbf{x}}^-(y) + \mathbf{K}(y)[\mathbf{z}(y) - \mathbf{h}(\hat{\mathbf{x}}^-(y))] \tag{23}$$

$$\hat{\mathbf{P}}^+(y) = \hat{\mathbf{P}}^-(y) - \mathbf{K}(y)\mathbf{J}_h(y)\hat{\mathbf{P}}^-(y) \tag{24}$$

4. The a priori estimates for the next state are predicted as

$$\hat{\mathbf{x}}^-(y+1) = \mathbf{F}\hat{\mathbf{x}}^+(y) \tag{25}$$

$$\mathbf{P}^-(y+1) = \mathbf{F}\mathbf{P}^+(y)\mathbf{F}^T \tag{26}$$

5. Repeat the above steps with the predicted estimates for the next state.

The recursive operation, as described above, is performed for each y $[1,N_s]$. For every run, the Kalman filter computes the state estimate using the prior value and measurement. To initialize the Kalman filter, we used $\hat{\mathbf{x}}^-(1) = [|\Gamma_i(1)|\text{angle}\{\Gamma_i(1)\}\ 0\ \ldots\ 0]^T$ and $\hat{\mathbf{P}}^-(1) = \mathrm{diag}(1,1,\ldots,1,1)$ as the prior estimates.

With the state estimate at $y = N_s$, i.e. $\hat{\mathbf{x}}^+(N_s)$ known, the polynomial coefficients are estimated from Eq. (8) as

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{bmatrix} = \mathbf{U}^{-1}(N_s) \begin{bmatrix} \hat{x}_2^+ \\ \hat{x}_3^+ \\ \hat{x}_4^+ \\ \vdots \\ \hat{x}_{M+2}^+ \end{bmatrix} \tag{27}$$

Finally, the phase $\phi_i(y)$ within the segment i is constructed from the estimated coefficients using Eq. (4). It needs to be emphasized that an unwrapped estimate of phase is directly obtained within the segment. The above procedure is repeated for all segments within a column, and subsequently for all columns to obtain the