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Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition

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ABSTRACT

This paper is concerned with the local and global existence of mild solution for an impulsive fractional functional integro differential equations with nonlocal condition. We establish a general framework to find the mild solutions for impulsive fractional integro-differential equations, which will provide an effective way to deal with such problems. The results are obtained by using the fixed point technique and solution operator on a complex Banach space.

1. Introduction

During the past decades, fractional differential equations have attracted many authors (see for instance [1–13] and references therein). This is mostly because they efficiently describe many phenomena arising in engineering, physics, economics, and science. Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Presently, fractional differential equations are considered as an alternative model to integer differential equations. For more details, one can see the monographs of [14–16].

The impulsive differential equations arising from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Such processes are naturally seen in biology, physics, engineering, etc. Due to their significance, many authors have been established the solvability of impulsive differential equations. For the general theory and applications of such equations we refer the interested reader to see the papers [1–3,6–8,12,13] and references therein.

Let X be a Banach space and $PC_t := PC([-r,t];X)$, $r > 0$, $0 \leqslant t \leqslant T < \infty$, be a Banach space of all such functions $\phi : [-r,t] \to X$, which are continuous every where except for a finite number of points t_i, $i = 1,\dots,m$, at which $\phi(t_i^+)$ and $\phi(t_i^-)$ exists and $\phi(t_i) = \phi(t_i^-)$, endowed with the norm

$$\|\phi\|_t = \sup_{-r\leqslant s\leqslant t} \|\phi(s)\|_X, \quad x \in PC_t,$$

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where $\|\cdot\|_X$ is the norm in X.

Now, consider the following impulsive fractional semilinear integro-differential equation with nonlocal initial condition:

$$\begin{cases} {}^cD_t^\alpha x(t) + Ax(t) = f(t, x_t) + \int_0^t q(t-s)g(s, x_s)ds, & t \in J = [0, T],\ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ h(x) = \phi_0 \quad \text{on } [-r, 0], \end{cases} \tag{1}$$

where ${}^cD_t^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative, $-A$ is sectorial operator. The nonlinear maps $f, g : J \times PC_0 \to X$, and $q : J \to X$, are continuous where $PC_0 = PC([-r, 0], X)$ and for any $x \in PC_T = PC([-r, T], X)$, $t \in J$, we denote by x_t the element of PC_0 defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. The function $\phi_0 \in PC_T$ and the map h is defined from PC_T into PC_T. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(X, X)$, $(k = 1, 2, \ldots, m)$, are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left-hand limits of $x(t)$ at $t = t_k$, respectively.

Rashid and Al-Omari [6] considered the following fractional order integro-differential equation in a Banach space X:

$$\begin{cases} u^\alpha(t) + Au(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, & t > 0,\ 0 < \alpha \leqslant 1, \\ u(0) = u_0 \in X, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, \ldots, m, \end{cases} \tag{2}$$

where $-A$ is assumed to be an infinitesimal generator of a compact semigroup $T(t)$, $t \geqslant 0$. Authors [6] have established the local and global existence of mild solutions for the considered problem using the fixed point technique. Due to the latest development regarding the mild solution for the considered problem of fractional order with impulsive conditions the definition of mild solution defined by the authors of [6] is not appropriate (see [5,26]).

In [17,18], authors introduced a more appropriate concept for mild solution of fractional finite and infinite time delay evolution systems and optimal controls in infinite dimensional spaces. The new introduced mild solution is associated with a probability density function and semigroup operator while in [5], authors defined mild solution for impulsive fractional differential equations which is associated with Mittag–Leffler function, solution operator and resolvent operator.

Recently, Feckan et al. in [8] give a counter example to show that the formula of solutions for impulsive fractional differential equations used in some previous papers are incorrect. Since the authors used that the Caputo derivative ${}^c{}_aD_t^\alpha$ restricted on $(a, b]$, $0 < a < b$, is ${}_a{}^cD_t^\alpha$, but unfortunately it does not hold. In [8], the authors introduced a correct formulation of solutions for an impulsive Cauchy problem with Caputo's fractional derivative. For more details of solutions for impulsive fractional differential equations, one can see the papers [7,19,20] and references therein.

Particular case of the problem (1) when $\alpha = 1$ has been considered by Bahuguna and Srivastava [25] and others. In [17], authors considered the system (1) with local initial condition and including control function in the absence of impulsive effects. Our work is motivated by the papers [6,8–10,26]. The problem considered by the authors of [6] becomes a particular case of the problem (1).

This paper is concerned with the local and global existence of mild solution for impulsive fractional functional integro-differential equations with nonlocal condition. We establish a general framework to find the mild solutions for impulsive fractional integro-differential equations. In this paper we define mild solution of (1) using the concept introduced in [5] in which the mild solution is associated with Mittag–Leffler function, solution operator and resolvent operator. Due to the presence of impulsive conditions, we also used the concept of mild solution introduced in [7]. The results are obtained by using the fixed point technique and solution operator on a complex Banach space. One example is presented to verify our results.

We organize the rest of this paper as follows: in Section 2, we present some necessary definitions, preliminary results and state the assumptions that will be used to prove our main results. The proof of local existence of mild solution is given in Section 3, in Section 4 we prove the global existence of mild solution and Section 5 contains an illustrative example.

2. Preliminaries

In this section, we shall introduce some basic definitions, properties and lemmas which are required to establish our results and the relevant references are given together with the definitions. The function spaces defined in the introduction section are same in rest of the paper and the notations for the function spaces have their usual meaning if it is not specified.

A two parameter function of the Mittag–Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_C \frac{\mu^{\alpha-\beta}e^\mu}{\mu^\alpha - z}d\mu, \quad \alpha, \beta > 0,\ z \in \mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leqslant |z|^{\frac{1}{2}}$ counter clockwise. The most interesting properties of the Mittag–Leffler functions are associated with their Laplace transform

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0,$$

see [14] for more details.