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Bursting mechanism in a time-delayed oscillator with slowly varying external forcing

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ABSTRACT

This paper investigates the generation of complex bursting patterns in the Duffing oscillator with time-delayed feedback. We present the bursting patterns, including symmetric fold–fold bursting and symmetric Hopf–Hopf bursting when periodic forcing changes slowly. We make an analysis of the system bifurcations and dynamics as a function of the delayed feedback and the periodic forcing. We calculate the conditions of fold bifurcation and Hopf bifurcation as well as its stability related to external forcing and delay. We also identify two regimes of bursting depending on the magnitude of the delay itself and the strength of time delayed coupling in the model. Our results show that the dynamics of bursters in delayed system are quite different from those in systems without any delay. In particular, delay can be used as a tuning parameter to modulate dynamics of bursting corresponding to the different type. Furthermore, we use transformed phase space analysis to explore the evolution details of the delayed bursting behavior. Also some numerical simulations are included to illustrate the validity of our study.

1. Introduction

The studies of bursting or mixed-mode oscillations have received great attention recently in modeling realistic neuronal networks, physics, mechanics and engineering systems and so on [1–5]. Bursting oscillations are waveforms that consist of alternating small and large amplitude excursions. Mathematically, the generation of bursting oscillations is often associated with fast and slow subsystems simultaneously [6,7]. Traditionally bursting oscillations can be created by the system switching between the coexisting attractors of the fast subsystem corresponding to the slow current. We call a system in down-state when variables exhibit small amplitude. Then the effect of two time scales may lead the system to up-state state, in which variables behave in large amplitude.

Most of the related papers focused on analyzing dynamics of bursting oscillations. Lewis and Rinzel [8] and Izhikevich [9] first gave a complete classification of different types of bursters. Their fast-slow dynamics analysis has become a well-accepted approach to study bursting. In addition, the geometrical bifurcation analysis is also used to investigate the generation or transition of bursting modes in the single neuron and bifurcation mechanism for synchronized bursting of two-cell systems [10–12]. Simo and Woafo [13] presented the bursting oscillations in a system consisting of a double-well magnetically coupled electrical oscillator. Shorten and Wall [14] studied bursting in a Hodgkin–Huxley type of neuronal model and its transition to other models. Bertram et al. [15] studied bursting behavior in the Chay-Cook model with two-parameter analysis. In Refs. [16,17], bifurcation mechanism of bursting has also been studied by the unfolding and normal form theory.

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These results attempt to propose methods to uncover the bifurcation mechanism of transition between down-state and up-state.

So far, the analysis and classification of bursting are mainly concentrated on systems without any delay. However, it is very useful that systems involve delays which can be in the description of the real world more explicitly [18,19]. Thus, time-delay has become an important tool for describing and controlling various nonlinear phenomena in numerous and diverse fields [20–25]. Although much is known about bursting phenomena and delayed systems, little is known about the relationship between them, as well as how time delay could affect bursting oscillations, the mechanism of which still need to be investigated.

Here we consider the well-known Duffing oscillator [26–28] with negative linear stiffness and linear time delay driven by a slowly periodically varying external forcing, expressed in the form of

$$\ddot{x} + \dot{x} - x + x^3 + Ax_\tau = k\sin(\omega t), \tag{1}$$

where $k > 0$ is the forcing amplitude, $0 < \omega \ll 1$ is the forcing frequency, $x_\tau = x(t-\tau)$ is a time delay. A means gain coefficient about the delay. If $A > 0$, it means a positive delayed feedback, and negative feedback if $A < 0$.

Because two time scales evolve in the vector field, bursting oscillations in different forms, can be observed. The dynamic behaviors and mechanism of the system are explained by local divergence. The paper is organized as follows: In Section 2, an analysis of the bifurcations and dynamics is obtained as a function of the linear delayed state feedback and the external periodic forcing. In Sections 3 and 4, bursting phenomena (symmetric fold/fold bursting, symmetric sup-Hopf/sup-Hopf bursting) and the effects of some parameters including time delay on different bursting are discussed. Investigations of occurrence and mechanism of certain bursting dynamics are also presented. Finally, Section 5 concludes the paper.

2. Bifurcation analysis

System (1) can be reduced to the non-delay system with external excitation at $\tau = 0$. For $\omega \ll 1$, the external excitation varies more slowly than the rest variables in the system. Periodic bursting and chaotic bursting, i.e. a series of spikes which are periodically or chaotically interspersed with the spiking period and quiescence behavior in multiscale systems have been studied more. Here, we concentrate on bursters under the condition of $\tau \neq 0$. In this case, the properties of bursting are influenced by the delay. At first, we shall analyze the bifurcation behavior of the equation of (1).

For $\omega \ll 1$, considering $k\sin(\omega t)$ as a control parameter δ, given by the following delay-differential equation (DDE):

$$\ddot{x} + \dot{x} - x + x^3 + Ax_\tau = \delta \tag{2}$$

System (2) can easily be converted into (3)

$$\begin{cases} \dot{x} = y \\ \dot{y} = \ddot{x} = -y + x - x^3 - Ax_\tau + \delta \end{cases} \tag{3}$$

Denoting the equilibrium points as (x_0, y_0), obviously $y_0 = 0$, x_0 is decided by the equation

$$x - x^3 - Ax + \delta = 0, \text{ i.e. } -x^3 + (1-A)x + \delta = 0 \tag{4}$$

The root discriminant of (4) is $\Delta = 81\delta^2 - 12(1-A)^3$, which gives the results: for $\Delta < 0$, there are three equilibrium points; for $\Delta = 0$, there are two equilibrium points; otherwise $\Delta > 0$, there is one and only equilibrium point in DDE. Particularly taking $A = 0$, $\delta = 0$, three equilibrium points are $E_0 = (0, 0)$ and $E_\pm = (\pm 1, 0)$. For the chosen values of the parameters, E_0 is a saddle and $E_\pm$ are stable focuses, implying the non-delayed system is bi-stable. To get a clear idea of the distribution of equilibrium points of DDE, we plot the curve of equilibrium points related to equation of (4) with respect to δ, for fixed values $A = 0.5$ and $A = -0.5$ in Fig. 1, while the equilibrium points for other values can be derived accordingly. One should be

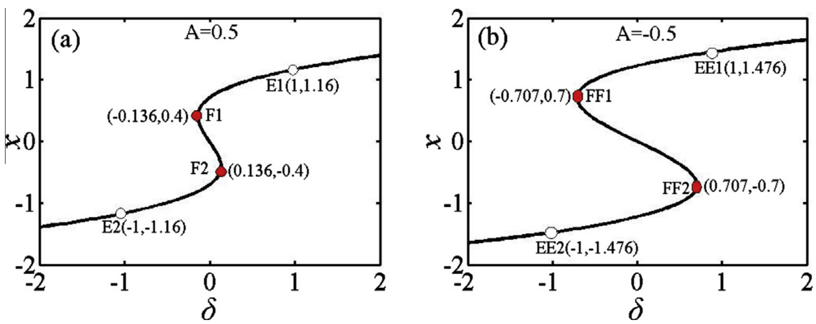

Fig. 1. Curve of equilibrium points in DDE related to Eq. (4) with respect to δ, solid points mean fold bifurcation points, and hollow points mean equilibrium points when $\delta = \pm 1$. (a) $A = 0.5$. (b) $A = -0.5$.