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Functional constraints method for constructing exact solutions to delay reaction–diffusion equations and more complex nonlinear equations

Andrei D. Polyanin [a,*,1], Alexei I. Zhurov [a,b,*]

[a] Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, Moscow 119526, Russia
[b] Cardiff University, Heath Park, Cardiff CF14 4XY, UK

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ABSTRACT

We propose a new method for constructing exact solutions to nonlinear delay reaction–diffusion equations of the form

$$u_t = ku_{xx} + F(u, w),$$

where $u = u(x,t), w = u(x, t-\tau)$, and τ is the delay time. The method is based on searching for solutions in the form $u = \sum_{n=1}^{N} \xi_n(x)\eta_n(t)$, where the functions $\xi_n(x)$ and $\eta_n(t)$ are determined from additional functional constraints (which are difference or functional equations) and the original delay partial differential equation. All of the equations considered contain one or two arbitrary functions of a single argument. We describe a considerable number of new exact generalized separable solutions and a few more complex solutions representing a nonlinear superposition of generalized separable and traveling wave solutions. All solutions involve free parameters (in some cases, infinitely many parameters) and so can be suitable for solving certain problems and testing approximate analytical and numerical methods for nonlinear delay PDEs. The results are extended to a wide class of nonlinear partial differential–difference equations involving arbitrary linear differential operators of any order with respect to the independent variables x and t (in particular, this class includes the nonlinear delay Klein–Gordon equation) as well as to some partial functional differential equations with time-varying delay.

1. Introduction

Nonlinear delay partial differential equations and systems of coupled equations arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, ecology, economics, and many other areas (e.g. see the studies [1–11] and references in them). It is noteworthy that similar equations occur in the mathematical theory of artificial neural networks, whose results are used for signal and image processing as well as in image recognition problems [12–21].

The present paper deals with nonlinear delay reaction–diffusion equations [1,3,11,22] of the form

$$u_t = ku_{xx} + F(u, w), \quad w = u(x, t-\tau). \tag{1}$$

A number of exact solutions to the heat equation with a nonlinear source, which is a special case of Eq. (1) without delay and with $F(u, w) = f(u)$, are listed, for example, in [23–29]. A comprehensive survey of exact solutions to this nonlinear equation

* Corresponding authors.
E-mail addresses: polyanin@ipmnet.ru (A.D. Polyanin), zhurovai@cardiff.ac.uk, zhurov@ipmnet.ru (A.I. Zhurov).
[1] Principal corresponding author.

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can be found in the handbook [30]; it also describes a considerable number of generalized and functional separable solutions to nonlinear reaction–diffusion systems of two coupled equations without delay.

The list of known exact solutions to Eq. (1) is quite limited.

In general, Eq. (1) admits traveling-wave solutions, $u = u(\alpha x + \beta t)$. Such solutions are dealt with in many studies (e.g. see the papers [2–7] and references in them). Simple separable solutions of Eq. (1) were studied in Ref. [22].

A complete group analysis of the non-linear differential–difference equation (1) was carried out in [11]. Four equations of the form (1) were found to admit invariant solutions; two of these equations are of limited interest, since they have degenerate solutions (linear in x). There was only one equation that involved an arbitrary function and had a non-degenerate solution:

$$u_t = ku_{xx} + u\left[-a\ln u + f(wu^{-b})\right], \quad b = e^{a\tau}, \tag{2}$$

where $f(z)$ is an arbitrary function. The exact solution to this equation found in [11] was

$$u = \exp(Cxe^{-at})\varphi(t), \tag{3}$$

where C is an arbitrary constant and $\varphi(t)$ is a function satisfying the delay ordinary differential equation

$$\varphi'(t) = \varphi(t)\Big[C^2ke^{-2at} - a\ln\varphi(t) + f(\varphi(t-\tau)\varphi^{-b}(t))\Big]. \tag{4}$$

The other equation obtained in Ref. [11] that had a non-degenerate solution coincides, up to notation, with a special case of Eq. (4), at $f(z) = c_1 + c_2\ln z$.

Remark 1. It is noteworthy that Eq. (2) is explicitly dependent on the delay time τ, which corresponds to a more general kinetic function $F(u, w, \tau)$ than in Eq. (1). It is only at $a = 0$ in (2) that we have a kinetic function explicitly independent of $\tau : F(u, w) = uf(w/u)$; in this case, (3) represents a separable solution, $u = e^{Cx}\varphi(t)$.

In what follows, the term 'exact solution' with regard to nonlinear partial differential–difference equations, including delay partial differential equations, is used in the following cases:

(i) the solution is expressible in terms of elementary functions or in closed form with definite or indefinite integrals;
(ii) the solution is expressible in terms of solutions to ordinary differential or ordinary differential–difference equations (or systems of such equations);
(iii) the solution is expressible in terms of solutions to linear partial differential equations.

Combinations of cases (i)–(iii) are also allowed.

This definition generalizes the notion of an exact solution used in Ref. [30] with regard to nonlinear partial differential equations.

Remark 2. Solution methods and various applications of linear and nonlinear ordinary differential–difference equations, which are much simpler than nonlinear partial differential–difference equations, can be found, for example, in Refs. [31–36].

Remark 3. For interesting biomedical and numerical applications of nonlinear delay partial differential equations and systems of such equations, see, for example, the studies [37–39] and references in them.

2. General description of the functional constraints method

Consider a wide class of nonlinear delay reaction–diffusion equations:

$$\begin{aligned} &u_t = ku_{xx} + uf(z) + wg(z) + h(z), \\ &w = u(x, t-\tau), \quad z = z(u, w), \end{aligned} \tag{5}$$

where $f(z), g(z)$, and $h(z)$ are arbitrary functions and $z = z(u, w)$ is a given function. In addition, we will sometimes consider more complex equations where f, g, and h can additionally depend on the independent variables x or/and t explicitly.

We look for generalized separable solutions of the form

$$u = \sum_{n=1}^{N}\Phi_n(x)\Psi_n(t), \tag{6}$$

where the functions $\Phi_n(x)$ and $\Psi_n(t)$ are to be determined in the analysis.

Remark 4. For nonlinear partial differential equations, various modifications of the method of generalized separation of variables based on searching for solutions of the form (6) are detailed, for example, in Ref. [27,28,30]. These studies also present a large number of equations that admits generalized separable solutions.