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Spectral numerical schemes for time-dependent convection with viscosity dependent on temperature

J. Curbelo [a,b], A.M. Mancho [a,*]

[a] Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UC3M), Nicolás Cabrera, 13-15, Campus Cantoblanco UAM, 28049 Madrid, Spain
[b] Departamento de Matemáticas de la Universidad Autónoma de Madrid, Facultad de Ciencias, Módulo 17, 28049 Madrid, Spain

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ABSTRACT

This article proposes spectral numerical methods to solve the time evolution of convection problems with viscosity strongly dependent on temperature at infinite Prandtl number. Although we verify the proposed techniques solely for viscosities that depend exponentially on temperature, the methods are extensible to other dependence laws. The set-up is a 2D domain with periodic boundary conditions along the horizontal coordinate which introduces a symmetry in the problem. This is the O(2) symmetry, which is particularly well described by spectral methods and motivates the use of these methods in this context. We examine the scope of our techniques by exploring transitions from stationary regimes towards time dependent regimes. At a given aspect ratio, stable stationary solutions become unstable through a Hopf bifurcation, after which the time-dependent regime is solved by the spectral techniques proposed in this article.

1. Introduction

Thermal convection in fluids in which viscosity depends on temperature plays an important role in many geophysical and technical processes. This problem is addressed in the literature by considering diverse laws. For instance, an Arrhenius-type viscosity law is a common approach to describing upper mantle convection problems [1–4]. Other studies such as [3, 5–8] consider fluids in which viscosity depends exponentially on temperature. In [6], an exponential law is chosen to fit the experimental data for the temperature dependence of viscosity in glycerol. In [3], the exponential dependence is discussed as an approach to the Arrhenius law by means of a Taylor series around a reference temperature. This is also called the Frank–Kamenetskii approximation (see [9]). In [8], extremely large viscosity variations such as those expected in the mantle are investigated by means of an exponential law. More recently, [10,11] have considered the hyperbolic tangent or the arctangent as viscosity laws since they model a viscosity transition in a narrow temperature gap. Further studies have treated other weaker dependencies such as linear [12,13] or quadratic ones [14,15].

From the mathematical point of view it has recently been proven that convection problems in which viscosity is a function of temperature is a well posed problem [16,17] for dependences which are smooth bounded positive analytical functions, so it stands a good chance of solution on a computer using a stable algorithm. The variability in viscosity introduces strong couplings between the momentum and heat equations, as well as introducing important nonlinearities into the whole problem. On the other hand, it is of particular interest for mantle convection problems, to consider the fact that the Prandtl number, which is the quotient of viscosity and thermal diffusivity, is virtually infinite. This limit transforms the set of equations describing the time-dependent problem into a differential algebraic problem (DAE), which is very stiff.

* Corresponding author. Tel.: +34 912999762.
E-mail addresses: jezabel.curbelo@icmat.es (J. Curbelo), a.m.mancho@icmat.es (A.M. Mancho).

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In this context, this article discusses the performance of several time-evolution spectral schemes for convection problems in which the viscosity depends on temperature and the Prandtl number is infinite. The analysis is focused on the choice of an exponential law similar to that discussed in [18]. We characterize time-dependent solutions demonstrating the efficiency of the time-dependent scheme to describe the solutions beyond the stationary regime. Our setting is a 2D domain with periodic boundary conditions along the horizontal coordinate. The equations with periodic boundary conditions are invariant under horizontal translations, thus the problem has a symmetry represented by the SO(2) group. Additionally, if the reflection symmetry exists, the full symmetry group is the O(2) group. Symmetric systems typically exhibit more complicated behavior than non-symmetric systems and there exist numerous novel dynamical phenomena whose presence is fundamentally related to the presence of symmetry, such as traveling waves or stable heteroclinic cycles [19–21]. The numerical simulation of dynamics under the presence of symmetry has usually been addressed by spectral techniques [21,22]. In the context of convection problems with constant viscosity in cylindrical containers that posses the O(2) symmetry, the existence of heteroclinic cycles have been reported both experimentally [23] and numerically with a fully spectral approach [24]. However, Assemat and co-authors [25], who use high-order finite element methods to solve a similar set-up, remark the absence of the heteroclinic cycles in their simulations. Additionally they detect the influence of the computational grid on the breaking of the O(2) symmetry by producing pinning effects on the solution. These findings suggest that spectral methods may be particularly suitable for addressing problems with symmetries since some solutions may be overlooked with approaches based on other spatial discretizations.

Spectral methods are not very popular in the simulation of convection problems with temperature-dependent viscosity [26], as they are reported to have limitations when handling lateral variations in viscosity. Alternatively, preferred schemes exist in which the basis functions are local; for example, finite difference, finite element and finite volume methods. For instance, the works by [27,28] have treated this problem in a finite element discretization in primitive variables, while in [29,30] finite differences or finite elements are used in the stream-function vorticity approach. Spectral methods have been successfully applied to model mantle convection with moderate viscosity variations in, for instance, [31,32]. These works do not use the primitive variables formulation and deal with the variations in viscosity by decomposing it into a mean (horizontally averaged) part and a fluctuating (laterally varying) part. Our approach addresses the variable viscosity problem by proposing a spectral approximation in primitive variables without any decomposition on the viscosity. The main novelty in this paper is the extension of the spectral methodology discussed in [18], valid only for stationary problems, for solving the time-dependent problem and the extension of the results to describe time-dependent solutions.

As regards temporal discretization, backward differentiation formulas (BDF's) are widely used in convection problems. This is the case of the work discussed in [33], which following ideas proposed in [34] uses a fixed time step second-order-accurate which combines Adams–Bashforth and BDF schemes. A recent article by García and co-authors [35] compares the performance of several semi-implicit and implicit time integrations methods based on BDF and extrapolation formulas. The physical set-ups discussed in these papers are for convection problems with constant viscosity and finite Prandtl number. In contrast, this article focuses on convection problems with viscosity strongly dependent on temperature and infinite Prandtl number that lead to a differential algebraic problem. We will see that the semi-implicit methods discussed in [33,34] do not work in this context. BDFs and implicit methods are known to be an appropriate choice [36,37] for efficiently tackling very stiff problems. According to [35,36], for the time discretization scheme we propose several high order backward differentiation formulas which are ready for an automatic stepsize adjustment. Furthermore, we solve the fully implicit problem and also propose a semi-implicit approach. The output and performance of this option are compared with those of the implicit scheme. It is found that the semi-implicit approach presents some advantages in terms of computational performance.

The article is organized as follows: In Section 2, we formulate the problem, providing a description of the physical set-up, the basic equations and the boundary conditions. Section 3 describes a spectral scheme for stationary solutions which will be useful for benchmarking the time dependent numerical schemes. First the conductive solution and its stability is determined. Other stationary solutions appear above the instability threshold, which are computed by means of a Newton–Raphson method using a collocation method. The stability of the stationary solutions is predicted by means of a linear stability analysis and is also solved with the spectral technique. Section 4 discusses several time-dependent schemes, which include implicit and semi-implicit schemes. Section 5 reports the results at a fixed aspect ratio in a range of Rayleigh numbers. Stationary and time-dependent solutions are found and different morphologies of the thermal plumes are described. Some computational advantages of some schemes versus others are discussed. Finally, Section 6 presents the conclusions.

2. Formulation of the problem

The physical set-up, shown in Fig. 1, consists of a two dimensional fluid layer of depth d (z coordinate) placed between two parallel plates of length L. The bottom plate is at temperature T_0 and the upper plate is at T_1, where $T_1 = T_0 - \Delta T$ and ΔT is the vertical temperature difference, which is positive, *i.e*, $T_1 < T_0$.

In the equations governing the system, $\mathbf{u} = (u_x, u_z)$ is the velocity field, T is the temperature, P is the pressure, x and z are the spatial coordinates and t is the time. Equations are simplified by taking into account the Boussinesq approximation, where the density ρ is considered constant everywhere except in the external forcing term, where a dependence on temperature is assumed as follows $\rho = \rho_0(1 - \alpha(T - T_1))$. Here ρ_0 is the mean density at temperature T_1 and α the thermal expansion coefficient.