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Physical limit of prediction for chaotic motion of three-body problem

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ABSTRACT

A half century ago, Lorenz found the "butterfly effect" of chaotic dynamic systems and made his famous claim that long-term prediction of chaos is impossible. However, the meaning of the "long-term" in his claim is not very clear. In this article, a new concept, i.e. the physical limit of prediction time, denoted by T_p^{max}, is put forwarded to provide us a time-scale for at most how long mathematically reliable (numerical) simulations of trajectories of a chaotic dynamic system are physically correct. A special case of three-body problem is used as an example to illustrate that, due to the inherent, physical uncertainty of initial positions in the (dimensionless) micro-level of 10^{-60}, the chaotic trajectories are essentially uncertain in physics after $t > T_p^{max}$, where $T_p^{max} \approx 810$ for this special case of the three body problem. Thus, physically, it has no sense to talk about the "accurate, deterministic prediction" of chaotic trajectories of the three body problem after $t > T_p^{max}$. In addition, our mathematically reliable simulations of the chaotic trajectories of the three bodies suggest that, due to the butterfly effect of chaotic dynamic systems, the micro-level physical uncertainty of initial conditions might transfer into macroscopic uncertainty. This suggests that micro-level uncertainty might be an origin of some macroscopic uncertainty. Besides, it might provide us a theoretical explanation about the origin of uncertainty (or randomness) of many macroscopic phenomena such as turbulent flows, the random distribution of stars in the universe, and so on.

1. Introduction

Why do stars in the sky look like random? Why are velocities of turbulent flows so uncertain? Are there any relationships between microscopic uncertainty and macroscopic uncertainty? What is the origin of the macroscopic uncertainty and/or randomness? Can we give a theoretical explanation for such kind of macroscopic uncertainty and/or randomness? In this paper, we attempt to give such an explanation using chaotic motion of Newtonian three-body problem as an example.

Making mathematically reliable and physically correct prediction is an important scientific object. Laplace claimed that the future can be unerringly predicted, given sufficient knowledge of the present. However, Laplace was wrong even in a classical, non-chaotic universe, as pointed out by Wolpert [1]. In 1890, Poincaré [2] found that trajectories of three-body problem are unintegrable in general. In 1963 Lorenz [3] found that it is impossible to make "long-term" prediction of non-periodic (called chaotic today) solution of a nonlinear dynamic system (called today Lorenz equation), due to the sensitive dependence on initial condition (SDIC), i.e. a very slight variation of initial conditions leads to considerably obvious difference of computer-generated trajectories for a long time. Besides, it was found [4–10] that numerical simulations of

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chaotic trajectories are sensitive not only to initial conditions but also to (traditional) numerical algorithms and digit precision of numerical data, i.e. a slight difference of (traditional) numerical algorithms (such as a tiny variation of time step) and/or digit precision of data might lead to considerably large difference of simulations of chaotic trajectories for a long time. This is easy to understand, since numerical noises, i.e. truncation and round-off errors, are inherent and unavoidable for all numerical simulations, where truncation error is closely related to numerical algorithms and round-off error is due to the limited precision of numerical data. According to the Shadow Lemma [11], for a uniformly hyperbolic dynamic system, there always exists a true trajectory near any computer-generated trajectories, as long as the truncation and round-off errors are small enough. Unfortunately, hardly a nonlinear dynamic system is uniformly hyperbolic in most cases so that the Shadow Lemma [11] often does not wok in practice. Besides, it is found that for some chaotic dynamic systems, computer-generated trajectories can be shadowed only for a short time [12], and in addition it is "virtually impossible to obtain a long trajectory that is even approximately correct" [13]. Furthermore, for some chaotic systems, "there is no fundamental reasons for computer-simulated long-time statistics to be even approximately correct" [14]. As illustrated by Yuan and York [15] using a model, a numerical artifact persists for an arbitrarily high numerical precision. These numerical artifacts "expose an exigent demand of safe numerical simulations" [16]. Thus, it is a huge challenge to give mathematically reliable simulations of chaotic dynamic systems in a long enough interval.

On the other side, although Lorenz's famous claim that "long-term prediction of chaos is impossible" has been widely accepted by scientific community, the word "long-term" is not very clear. Is one day long enough? Or millions of years? Obviously, a time-scale is needed for us to understand Lorenz's claim better.

Currently, a safe numerical approach, namely the "Clean Numerical Simulation" (CNS) [10,17], is proposed to gain mathematically reliable computer-generated trajectories of chaotic dynamic system in a finite but long enough interval. The CNS is based on Taylor series method (TSM) [18,19] at high enough order of approximation and the high precision data with large enough number of significant digits. The Taylor series method [18,19] is one of the oldest method, which can trace back to Newton, Euler, Liouville and Cauchy. It has an advantage that its formula at arbitrarily high order can be easily expressed in the same form. So, from viewpoint of numerical simulations, it is rather easy to use the Taylor series method at very high order so as to deduce the truncation error to a required level. Besides, the round-off error can be reduced to arbitrary level by means of computer algebra system (such as Mathematica) or the multiple precision (MP) library [20]. Thus, using the CNS, the numerical noises can be decreased to such a small level that both truncation and round-off errors are negligible in a given finite but long enough interval. For example, using the CNS ($\Delta t = 0.01$) with the 400th-order Taylor series method and 800-digit precision data, Liao [9] gained, for the first time, the mathematically reliable chaotic simulation of Lorenz equation in the interval $0 \leqslant t \leqslant 1000$ LTU (Lorenz time unit) by means of Mathematica. In 2011, Wang et al. [21] greatly decreased the required CPU times of the CNS by employing a parallel algorithm and the multiple precision (MP) library of C. Using the parallel CNS with the 1000th-order Taylor expansion and the 2100 digit multiple precision, Wang et al. [21] gain a mathematically reliable chaotic simulation of Lorenz equation in the interval [0,2500] within only 30 h, which is validated using a more accurate simulation given by the 1200th-order Taylor expansion and the 2100 digit multiple precision. Their results [21] confirm the correction and reliability of Liao's simulation [9] in the interval $[0, 1000]$. Currently, using 1200 CPUs of the National Supercomputer TH-A1 and the modified parallel integral algorithm based on the CNS with the 3500th-order Taylor expansion and the 4180-digit multiple precision data, Liao and Wang [22] gain a mathematically reliable simulation of chaotic solution of Lorenz equation in a rather long interval $[0, 10000]$. All of these illustrate that, the uncertainty of simulations of chaotic trajectories (in a given, finite but long enough interval) caused by numerical noises can be avoided by means of the CNS. Thus, from *mathematical* viewpoint, given an *exact* initial condition, we can gain *mathematically* reliable trajectories of chaotic dynamic systems in a *finite* but long *enough* interval by means of the CNS, without any observable uncertainty of simulation.

Is such a *mathematically* reliable chaotic trajectory (in a finite but long enough interval) *physically* correct?

Note that the uncertainty of simulations of chaotic trajectories is caused by many factors. Theoretically speaking, given an *exact* initial condition, the uncertainty is completely caused by numerical noises, i.e. truncation and round-off error, where truncation error is determined by numerical algorithms and round-off error is due to the limited precision of numerical data, respectively. However, in practice, initial conditions are *not* exact in practice: they contain both artificial and physical uncertainty. The artificial uncertainty mainly comes from limited precision of measurement. The physical uncertainty is due to the inherently uncertain/random property of nature, caused by such as thermal fluctuation, wave-particle duality of de Broglie's wave, and so on. Generally, the artificial uncertainty is much larger than the physical uncertainty. So, for dynamic systems, physical uncertainty determines a time of physical limit of prediction, denoted by T_p^{max}, beyond which trajectories are essentially uncertain in *physics*, as illustrated in this paper. In order to investigate the time of physical limit of prediction, we *assume* that there is no artificial uncertainty, i.e. measurement can have precision of arbitrary degree. In this way, the uncertainty of initial conditions caused by limited precision of measurement is avoided, too. Thus, we can focus on the case that initial conditions contain micro-level physical uncertainty only, which are often in the (dimensionless) micro-level of 10^{-20} to 10^{-60} or even smaller. It should be emphasized that the uncertainty of initial condition is in the micro-level, but the considered dynamic system is about macroscopic phenomena. Thus, this is a problem with multiple scales. Fortunately, the propagation of such kind of micro-level uncertainty can be reliably and accurately simulated by means of the CNS now, as illustrated by Liao [17].

Without loss of generality, we consider here the famous three-body problem [2,24,23,25] with chaotic motion, governed by Newtonian gravitational law. In Section 2 we briefly describe the numerical algorithms based on the CNS approach.