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Differentiability of type-2 fuzzy number-valued functions

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ABSTRACT

In this paper, we define a differentiability of the type-2 fuzzy number-valued functions. The definition is based on type-2 Hukuhara difference which is defined in the paper as well. The related theorem of the differentiability of the type-2 fuzzy number-valued functions is derived. In addition, a parametric closed form of the perfect triangular quasi type-2 fuzzy numbers is introduced, and finally, the applicability and an approach to solving type-2 fuzzy differential equations are illustrated using some examples and cases.

1. Introduction

Following the introduction of conventional Fuzzy Sets [FSs] in 1965 – presently known as Type-1 Fuzzy Sets [T1FSs] – Zadeh presented a more general concept under the title "Type-2 Fuzzy Sets" [T2FSs] in 1975 [1]. By definition, T2FS is a set in which membership grades are a T1FS each. As a matter of fact, a T2FS includes not only data uncertainty, but also how the Membership Function [MF] uncertainty is presented. As a way of illustration, suppose a number of people are asked about the temperature of the room where they are present. All the subjects mention "approximately 70° Fahrenheit". Nonetheless, if each individual subject is asked to show the "approximately 70° Fahrenheit MF", different MFs are likely to be presented, even if the MFs are all of the same kind (e.g., triangular). This implies the statement that "Words can mean different things to different people" [2]. Accordingly, T2FSs prove helpful in cases where an exact form of an MF cannot be determined.

Thanks to the uncertainty in modeling many dynamic systems, using FSs as an effective tool has attracted much attention. The question that arises here is what type of FS is a better alternative in modeling. This might not sound an easy question to answer, depending on different problem conditions. On the one hand, in spite of the fact that some useful modeling and applications applying T1FSs have been achieved (see e.g., [3–6]), T1FSs have limited the capability to directly model and minimize the effect of data uncertainty [2]. On the other hand, there are so many problems, like the mentioned room temperature case, in which the exact form of MFs cannot be determined. It follows that T2FSs had better be applied which means higher costs that are involved, namely more complexity and computation. Let then, the selection of the appropriate type of FSs be eventually left to the decision makers.

Since the modeling of quite a few dynamic systems could be done using differential equations, and because naturally there is uncertainty in data and/or parameters, for considering the existing uncertainty, examining differential equations

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based on FSs – Fuzzy Differential Equations [FDEs] – seems quite essential. As a result, in recent years, FDEs have been widely investigated worldwide.

To define an FDE, the differentiability of the fuzzy function should be initially defined. The concept of fuzzy derivative first introduced by Zadeh and Chang [7] in 1972. Since then, numerous definitions of the differentiability of fuzzy functions have been presented. Among these, the Hukuhara differentiability [*H-differentiability*] and the strongly generalized differentiability [8,9], have attracted more attention. Therefore, in recent literature, many papers about FDEs are found using H-differentiability and the strongly generalized differentiability (see e.g., [10,11]).

Because the definitions of the differentiability of fuzzy functions have been done based T1FSs, this paper aims at defining the differentiability of fuzzy functions based on T2FSs. To the best of our knowledge, the present work is the first time in the literature that the differentiability of the type-2 fuzzy number-valued functions and the differential equations relevant to the type-2 fuzzy number-valued functions are considered.

Let differential equations that are defined based on T1FSs be named Type-1 Fuzzy Differential Equations [T1FDEs], and differential equations that are defined based on T2FSs be named Type-2 Fuzzy Differential Equations [T2FDEs].

In this paper, we first defined the Hukuhara difference [*H-difference*] based on perfect Type-2 Fuzzy Numbers [T2FNs] defined by Hamrawi and Coupland [12,13], and then, using the $\tilde{\alpha}-plane$ representation, we proved that the Type-2 Hukuhara difference [H_2-*difference*] reduces to *H-difference*. Then, triangular perfect Quasi Type-2 Fuzzy Numbers [QT2FNs] [12] are presented in a parametric closed form. Using a distance metric between T2FSs [14] and the perfect T2FNs, metric space was formed. Above all, we define the differentiability of the type-2 fuzzy number-valued functions based on H_2-*difference*, and derive a theorem related to the differentiability of the type-2 fuzzy number-valued functions. Eventually, the applicability and an approach to solving T2FDEs are illustrated using two examples and cases.

This paper is organized as follows: Section 2 gives some basic concepts and definitions. Section 3, contains definitions of T2FNs and the parametric closed form of the triangular perfect QT2FNs. In Section 4, H_2-*difference* and the differentiability of the type-2 fuzzy number-valued functions are defined, and the related theorems are derived. Section 5 presents some examples and cases of T2FDEs in modeling and shows how to obtain their solutions. Consequently, conclusions are discussed in Section 6.

2. Basic concepts

Throughout this paper, the set of all real numbers is denoted by $\mathbb{R}$, the set of all Type-1 Fuzzy Numbers [T1FNs] on $\mathbb{R}$ by E_1 and the set of all perfect T2FNs on $\mathbb{R}$ by E_2. The $\alpha-cut$ of a fuzzy set A is denoted by A^α.

Definition 2.1. A type-1 fuzzy set $u(x) \in E_1, u : \mathbb{R} \to [0,1]$ is a T1FN if it satisfies the following requirements:

(a) $u(x)$ is normal i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$.
(b) $u(x)$ is fuzzy convex i.e., $\forall x_1, x_2 \in \mathbb{R}, \lambda \in [0,1] \quad u(\lambda x_1 + (1-\lambda)x_2) \geqslant \min(u(x_1), u(x_2))$.
(c) $\operatorname{supp} u(x) = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the $u(x)$, and its closure $cl(\operatorname{supp} u(x))$ is compact.
(d) $u(x)$ is upper semi-continuous.

Definition 2.2. Let $u, v \in E_1$. If there exists a $w \in E_1$ such that $u = v + w$, then w is called Hukuhara difference of u and v; and is denoted by $u \overset{H}{-} v$.

Definition 2.3 [9]. Let $f : (a,b) \to E_1$ and $t_0 \in (a,b) \subset \mathbb{R}$, then $f(t)$ is differentiable at t_0, in the first form, if there exists an element $f'(t_0) \in E_1$, such that for all $h > 0$, sufficiently near zero, there are $f(t_0+h) \overset{H}{-} f(t_0)$, $f(t_0) \overset{H}{-} f(t_0-h)$, as well as the following limits:

$$\lim_{h\to 0} \frac{f(t_0+h) \overset{H}{-} f(t_0)}{h} = \lim_{h\to 0} \frac{f(t_0) \overset{H}{-} f(t_0-h)}{h} = f'(t_0) \tag{2.1}$$

Or, $f(t)$ is differentiable at t_0, in the second form, if there exists an element $f'(t_0) \in E_1$, such that for all $h > 0$, sufficiently near zero, there are $f(t_0) \overset{H}{-} f(t_0+h)$, $f(t_0-h) \overset{H}{-} f(t_0)$, as well as the following limits:

$$\lim_{h\to 0} \frac{f(t_0) \overset{H}{-} f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h) \overset{H}{-} f(t_0)}{-h} = f'(t_0) \tag{2.2}$$

Here the limits are taken in the metric space (E_1, d_1) where $d_1(u,v) = \sup\{d_H(u^\alpha, v^\alpha) : 0 \leqslant \alpha \leqslant 1, u, v \in E_1\}$ and d_H is the Hausdorff distance.

Definition 2.4 [15]. The *Point-valued representation*: A T2FS, $\tilde{A}$, can be characterized by its type-2 membership function, $\mu_{\tilde{A}}(x,u)$, as: