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Asymptotic formulae of Liouville–Green type for higher even-order equations

A.S.A. Al-Hammadi

Department of Mathematics, College of Science, University of Bahrain, P.O. Box: 32038, Bahrain

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Abstract

Asymptotic formulae of Liouville–Green type for general linear ordinary differential equations of an arbitrary even-order $2m$ are investigated. A theorem on asymptotic behaviour at the infinity of $2m$ linearly independent solutions is proved. It is shown that numerous results known in the literature are contained in this theorem as particular cases.

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1. Introduction

We investigate the asymptotic form of ($2m$) linearly independent solutions for the even-order differential equation

$$(p_0y^{(m)})^{(m)}+\sum_{j=0}^{m-1}\left[\frac{1}{2}\left\{(q_{m-j}y^{(j)})^{(j+1)}+(q_{m-j}y^{(j+1)})^{(j)}\right\}+(p_{m-j}y^{(j)})^{(j)}\right]=0, \tag{1}$$

as $x\to\infty$, where x is the independent variable. We use the usual notation $y^{(j)}=\mathrm{d}^jy/\mathrm{d}x^j$, in particular the prime will denote $\mathrm{d}/\mathrm{d}x$. The functions $p_j(0\leqslant j\leqslant m)$ and $q_j(l\leqslant j\leqslant m)$ are defined on

E-mail address: profmaths_alhammadi@hotmail.com (A.S.A. Al-Hammadi).

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an interval $[a,\infty)$ and are not necessarily real-valued, while p_0 and p_m are nowhere zero in this interval. The Sturm–Liouville equation

$$(p_0y')' + p_1y = 0 \tag{2}$$

is a special case of Eq. (1). It has solutions given by the Liouville–Green asymptotic formula at $x\to\infty$:

$$y_k \sim (p_0p_1)^{-\frac{1}{4}}\exp\left(\pm\,\mathrm{i}\int_a^x\left(\frac{p_1}{p_0}\right)^{\frac{1}{2}}\mathrm{d}t\right). \tag{3}$$

Furthermore, if $p_1 = p_2 = \cdots = p_{m-1} = 0$ and $q_1 = q_2 = \cdots = q_m = 0$ in (1), then Eq. (1) reduces to

$$(p_0y^{(m)})^{(m)} + p_my = 0. \tag{4}$$

This even-order equation was considered by Hinton [1] who showed that at $x\to\infty$ Eq. (4) has solutions

$$y_k \sim (p_0p_m^{2m-1})^{-\frac{1}{4m}}\exp\left(w_k\int_a^x\left(\frac{p_m}{p_0}\right)^{\frac{1}{2m}}\mathrm{d}t\right), \tag{5}$$

where $w_k(1\leqslant k\leqslant 2m)$ are the $(2m)$th roots of (-1).

The formula (5) extends the form (3). Eastham [2] considered the nth order differential equation

$$(r_{n-1}\cdots(r_2(r_1y')')'\cdots)' + qy = 0. \tag{6}$$

He showed that (6) has solutions $y_k(1\leqslant k\leqslant n)$ as $x\to\infty$, with asymptotic forms

$$y_k \sim Q^{-\frac{(n-2)}{2}}(r_1^{n-1}r_2^{n-2}\cdots r_{n-2}^2r_{n-1})^{-\frac{1}{n}}\exp\left(w_k\int_a^x Q(t)\mathrm{d}t\right), \tag{7}$$

where $w_k(1\leqslant k\leqslant n)$ are the nth roots of (-1) with

$$Q = \{q/r_1r_2\cdots r_{n-1}\}^{\frac{1}{n}}. \tag{8}$$

We will see in Section 5 that (3), (5) and (7) are contained in our results as special cases. In this paper we use the recent asymptotic theorem of Eastham (see Section 2 of [4]) to obtain the solutions of (1). General features of our method are given in Sections 2 and 3. The main result for (1) is given in Section 4. Section 5 contains some comments.

2. The general method

We write (1) in the standard form of a first-order system used in [6]:

$$Y' = AY, \tag{9}$$

where y appears as the first component of Y, and $A(x)$ is the $n\times n$ matrix whose entries $a_{ij}(x)$ are given by