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Discrepancy principle for the dynamical systems method

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Abstract

Assume that
$Au = f$
is a solvable linear equation in a Hilbert space, $\|A\| < \infty$, and $R(A)$ is not closed, so this problem is ill-posed. Here $R(A)$ is the range of the linear operator A. A dynamical systems method for solving this problem, consists of solving the following Cauchy problem:

$$\dot{u} = -u + (B + \epsilon(t))^{-1}A^*f, \quad u(0) = u_0,$$

where $B := A^*A$, $\dot{u} := du/dt$, u_0 is arbitrary, and $\epsilon(t) > 0$ is a continuously differentiable function, monotonically decaying to zero as $t \to \infty$. Ramm has proved [Commun Nonlin Sci Numer Simul 9(4) (2004) 383] that, for any u_0, the Cauchy problem has a unique solution for all $t > 0$, there exists $y := w(\infty) := \lim_{t\to\infty} u(t)$, $Ay = f$, and y is the unique minimal-norm solution to $Au = f$. If f_δ is given, such that $\|f - f_\delta\| \leqslant \delta$, then $u_\delta(t)$ is defined as the solution to the Cauchy problem with f replaced by f_δ. The stopping time is defined as a number t_δ such that $\lim_{\delta\to 0}\|u_\delta(t_\delta) - y\| = 0$ and $\lim_{\delta\to 0} t_\delta = \infty$. A discrepancy principle is proposed and proved in this paper. This principle yields t_δ as the unique solution to the equation:

$$\|A(B + \epsilon(t))^{-1}A^*f_\delta - f_\delta\| = \delta,$$

where it is assumed that $\|f_\delta\| > \delta$ and $f_\delta \perp N(A^*)$. The last assumption is removed, and if it does not hold, then the right-hand side of the above equation is replaced by $C\delta$, where $C = \text{const} > 1$, and one assumes that $\|f_\delta\| > C\delta$. For nonlinear monotone A a discrepancy principle is formulated and justified.

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1. Introduction and statement of the result

Assume that

$$Au = f \tag{1}$$

is a solvable linear equation in a Hilbert space, $\|A\| < \infty$, and $R(A)$ is not closed, so problem (1) is ill-posed. Here $R(A)$ is the range of the linear operator A. Without loss of generality, assume that $\|A\| \leqslant 1$. Let y be the unique minimal-norm solution to (1). A solvable equation (1) is equivalent to

$$Bu = A^*f, \qquad B := A^*A, \tag{2}$$

where A^* is the operator adjoint to A. One has $N(B) = N(A) := N := \{v : Av = 0\}$. Let $Q := AA^*$, and $a > 0$ be a number. Then $\|B\| \leqslant 1$, $\|Q\| \leqslant 1$, and $(B + a)^{-1}A^* = A^*(Q + a)^{-1}$, as one easily checks. Denote by E_λ and F_λ the resolutions of the identity of B and Q, respectively.

Let $\epsilon(t)$ be a monotone, decreasing function,

$$\epsilon(t) > 0, \quad \lim_{t\to 0} \epsilon(t) = 0, \quad \lim_{t\to\infty} \sup_{\frac{t}{2}\leqslant s\leqslant t} |\dot\epsilon(s)|\epsilon^{-2}(t) = 0.$$

A dynamical systems method (DSM) for solving (1), consists of solving the following Cauchy problem:

$$\dot u = -u + (B + \epsilon(t))^{-1}A^*f, \quad u(0) = u_0, \ \dot u := \frac{du}{dt}, \tag{3}$$

where u_0 is arbitrary, and proving that, for any u_0, problem (3) has a unique solution for all $t > 0$, there exists $y := u(\infty) := \lim_{t\to\infty} u(t)$, $Ay = f$, and y is the unique minimal-norm solution to (1). These results are proved in [1]. If f_δ is given, such that $\|f - f_\delta\| \leqslant \delta$, then $u_\delta(t)$ is defined as the solution to (3) with f replaced by f_δ. The stopping time is defined as a number t_δ such that $\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0$, and $\lim_{\delta\to 0} t_\delta = \infty$. A discrepancy principle for choosing t_δ is proposed and proved in this paper.

Let us assume $f_\delta \perp N(A^*)$. Then this discrepancy principle, yields t_δ as the unique solution to the equation

$$\|A(B + \epsilon(t))^{-1}A^*f_\delta - f_\delta\| = \delta, \quad \|f_\delta\| > \delta, \tag{4}$$

and we prove that

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0, \quad \lim_{\delta\to 0} t_\delta = \infty. \tag{5}$$

The basic results of this paper are formulated in Theorems 1.1 and 2.1. In Example 1 we construct a monotone, linear, non-injective operator A, for which discrepancy principle (7), with $u_{\delta,\epsilon}$ solving equation $A(u_{\delta,\epsilon}) + \epsilon u_{\delta,\epsilon} = f_\delta$, does not yield convergence to the minimal-norm solution of Eq. (1).

Theorem 1.1. *If A is a bounded linear operator in a Hilbert space H, Eq.* (1) *is solvable*, $\|f_\delta\| > \delta$, $f_\delta \perp N(A^*)$, *and $\epsilon(t)$ satisfies the assumptions stated above, then Eq.* (4) *has a unique solution t_δ, and* (5) *holds, where y is the unique minimal-norm solution to* (1).