Available online at www.sciencedirect.com
SCIENCE DIRECT®

Mechanics Research Communications 32 (2005) 525–535

MECHANICS
RESEARCH COMMUNICATIONS

www.elsevier.com/locate/mechrescom

Multifield hyperelasticity: variational theorems for complex bodies

Marco Mosconi *

Istituto di Scienza e Tecnica delle Costruzioni, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

Available online 5 February 2005

Abstract

We present some multifield variational principles for complex elastic solids characterized by any material microstructure. Some general constitutive arguments in the context of multifield hyperelasticity are also discussed.

Keywords: Continua with microstructure; Multifield elasticity; Stored energy; Complementary stored energy; Variational principle

1. Introduction: continua with microstructure

In 'complex' solids the mechanical behavior under external actions is influenced by the material *microstructure*, i.e. the underlying heterogeneous material texture which has a typical characteristic length scale smaller than the body characteristic dimension.

Microstructures are observed in composites and nanocomposites (solids with embedded microscopic elements), porous and cracked solids, ceramics, polycrystalline materials, ferroelectrics and ferromagnetics, rocks and biological materials such as bones. Mechanical models of these bodies, resulting in a high predictable thoroughness, are of fundamental importance for the applications of these materials especially in new fields like bioengineering and nanotechnology, and the classical model of continuum mechanics must be enriched in order to account for the microstructure characterizing real bodies.

Multifield theories of complex solids rest upon the definition, besides the placement field x of classical continuum mechanics, of further kinematic descriptors, the *order parameters*, collecting the information pertaining to the observed substructure and treated as *observable* quantities (the set of order parameters is the 'microstate'; cf. Capriz, 1985). Accordingly, new measures of interaction arise behind the classical

* Tel.: +39 71 220 4553; fax: +39 71 220 4576.
E-mail address: mmosconi@univpm.it

0093-6413/$ - see front matter
doi:10.1016/j.mechrescom.2005.01.001

Cauchy stress, namely, the microstress and the internal self-force, which develop explicit mechanical power and need to be opportunely balanced.

Let $\mathfrak{B}$ be a complex body, occupying, in the Euclidean three-dimensional space, a regular region $\mathfrak{B}$, with well-defined outward unit normal n, chosen as reference configuration. Following Capriz (1985, 1989, 2000) and Mariano (2001), the *complete placement* of such a body (i.e. its current morphology) is given by the mapping $\mathfrak{B} \ni X \overset{k}{\mapsto} (x(X), \upsilon(X))$, with the placement field $\mathfrak{B} \ni X \overset{k_x}{\mapsto} x(X) \in \mathbb{R}^3$ giving the current place of each material element X and the order parameter field $\mathfrak{B} \ni X \overset{k_\upsilon}{\mapsto} \upsilon(X) \in \mathfrak{M}$ being a coarse grained descriptor of the material microstructure presented in X. Here $\mathfrak{M}$ is the microstructural manifold, i.e. the collection of all possible configurations assumed by the microstructure, considered as a differentiable paracompact manifold without boundary with finite dimension $\mathtt{m}$. We assume as customary that: $x(\mathfrak{B})$ is a regular region having a surface-like boundary, with outward unit normal well defined to within a finite number of corners and edges; the deformation gradient

$$F = \nabla x \tag{1.1}$$

has a positive determinant, $\det F > 0$, which guarantees that k_x is orientation preserving; $x(\cdot)$ and $\upsilon(\cdot)$ are both continuous and piecewise continuously differentiable on $\mathfrak{B}$.

Let us introduce the displacement field u defined by $x(X) = X + u(X)$; it results:

$$F = I + \nabla u \tag{1.2}$$

The nonlinear strain measure $D = \frac{1}{2}(F^{\mathrm{T}}F - I)$ is given by

$$D = \operatorname{sym} \nabla u + \frac{1}{2}(\nabla u)^{\mathrm{T}} \nabla u \tag{1.3}$$

in terms of the standard displacement u only, when the material microstructure is such that the order parameter does not appear in the deformation measure definitions. This is not the case, for instance, of micromorphic and Cosserat continua (cf. Capriz, 1989, §22–23).

Granted the possibility of defining a covariant gradient of the order parameter field, we introduce the *order parameter gradient*

$$G = \nabla \upsilon \tag{1.4}$$

which constitutes an essential ingredient to describe the morphology of a complex body.

For convenience, we suppose that the boundary $\partial\mathfrak{B}$ of the material body can be composed in the following manner:

$$\partial_u\mathfrak{B} \cup \partial_s\mathfrak{B} = \partial\mathfrak{B} \quad \text{with } \partial_u\mathfrak{B} \cap \partial_s\mathfrak{B} = \emptyset$$

$$\partial_\upsilon\mathfrak{B} \cup \partial_\tau\mathfrak{B} = \partial\mathfrak{B} \quad \text{with } \partial_\upsilon\mathfrak{B} \cap \partial_\tau\mathfrak{B} = \emptyset$$

and that the following kinematical boundary conditions, concerning both the placement and the order parameter field, can be applied:

$$u = u_0 \quad \text{on } \partial_u\mathfrak{B}, \quad \upsilon = \upsilon_0 \quad \text{on } \partial_\upsilon\mathfrak{B} \tag{1.5}$$

with $u_0 = x_0 - X$, for $X \in \partial_u\mathfrak{B}$, x_0 being a given field assigned on $\partial_u\mathfrak{B}$.

It is clear that condition $(1.5)_2$ depends intimately on the particular physical meaning attributed to the order parameter. In porous bodies υ is scalar-valued and represents the void volume fraction or porosity. In Cosserat bodies, υ is a rotation tensor or its associated vector. In micromorphic continua the order parameter is a second-order tensor with positive determinant (the gradient of micro-deformations; see Mindlin, 1964). In microcracked solids υ can be assumed as the tensorial approximation of the crack density distribution (see Mariano and Augusti, 1998). In ferroelectric solids, mentioning a last example amongst many