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An equivalent nonlinearization method for strongly nonlinear oscillations

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Abstract

An equivalent nonlinearization method is proposed for the study of certain kinds of strongly nonlinear oscillators. This method is to express the nonlinear restored force of an oscillatory system by a polynomial of degree two or three such that the asymptotic solutions can be derived in terms of elliptic functions. The least squares method is used to determine the coefficients of approximate polynomials. The advantage of present method is that it is valid for relatively large oscillations. As an application, a strongly nonlinear oscillator with slowly varying parameters resulted from free-electron laser is studied in detail. Comparisons are made with other methods to assess the accuracy of the present method.

Keywords: Strongly nonlinear oscillator; Equivalent nonlinearization; Slowly varying parameter; Least squares method; Free-electron laser

1. Introduction

In control engineering and oscillatory problems, we often meet with the following strongly nonlinear oscillator

$$\frac{d^2y}{dt^2} + \varepsilon k(y,\tilde{t})\frac{dy}{dt} + g(y,\tilde{t}) = 0 \tag{1}$$

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where $\tilde{t} = \varepsilon t (0 < \varepsilon \ll 1)$ is the slow scale. Methods of multiple scales and generalized KBM are effective to deal with such systems (for examples, Kuzmak, 1959; Kevorkian, 1987; Yuste, 1991; Dai and Zhuang, 1986; Cai and Li, 2004). However, these methods are based on the fact that the reduced equation ($\varepsilon = 0$), although nonlinear, has a solution in terms of known functions. The solution can be expressed in terms of Jacobian elliptic functions when $g(y, \tilde{t})$ is a polynomial of degree two or three with respect to y. For generally nonlinear functions $g(y, \tilde{t})$, Taylor series expansions (Nayfeh and Mook, 1979) and equivalent linearization methods (Krylov and Bogbliubov, 1943) are often used to approximate the nonlinear functions but the two methods are effective only for small amplitudes. Many efforts have been done to overcome this difficulty, such as Fourier series and approximate potential methods (Kevorkian and Li, 1988; Cai and Li, 2003), energy method (Li, 1995), generalized harmonic function (Xu and Cheung, 1994), perturbation-incremental method (Chan et al., 1996), combined equivalent linearization and averaging method (Mickens, 2003) and linearized perturbation technique (He, 2003). In this paper, an equivalent nonlinearization method is proposed to overcome the difficulty caused by certain kinds of nonlinear functions $g(y, \tilde{t})$. This method is to approximate the nonlinear function by a polynomial of degree two or three such that the leading approximation is expressible in terms of elliptic functions. The least squares method is used to determined the coefficients of approximate polynomials. As an application, a strongly nonlinear oscillator with slowly varying parameters which models motion of free-electron laser is studied in detail. Compared with Taylor series expansions method, the advantage of present method is that it is valid for relatively large oscillations. Comparisons are also made with the numerical method and Taylor series expansions method to show the efficiency of the present method.

2. Basic idea of equivalent nonlinearization method

For simplicity, we use the reduced equation of Eq. (1)

$$\frac{d^2y}{dt^2} + g(y) = 0 \tag{2}$$

to illustrate the main idea of equivalent nonlinearization method. We may seek a polynomial $p(y)$ to approximate the nonlinear function $g(y)$ if the characteristic of $g(y)$ is similar to a polynomial of degree two or three. The least squares method can be used to identify the coefficients of $p(y)$, which requires minimizing the following expression

$$\int_{y_1}^{y_2} (g(y) - p(y))^2 dy \to \min .$$

where y_1 and y_2 are chosen by the concerned range of oscillation. The potential energy $V(y) = \int_0^y g(u)du$ can give useful information about this. The details are illustrated by some examples.

Example 1. Consider the following nonlinear oscillator

$$\frac{d^2y}{dt^2} + a \sin y - by = 0 \tag{3}$$

The energy integral is

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + V(y) = 0$$

where

$$V(y) = -a \cos y - \frac{1}{2} by^2 + a$$