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# Symmetry group analysis of Benney system and an application for shallow-water equations

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**Abstract**

In this study we analyze symmetry group properties of the Benney system in the Eulerian description, which is in the form of the system of the coupled nonlinear integro-differential equations. We, first, find symmetry groups and obtain reduced forms, and then seek some similarity solutions to the reduced forms of the Benney equations. In addition, it is shown that one may transform solutions of the reduced forms of the Benney system into solutions of the reduced form of the one-dimensional shallow-water equations.


*Keywords:* Benney equations; Symmetry groups; Integro-differential equations; Shallow-water equations; Similarity solutions; Lie point symmetries

## 1. Introduction

The classical symmetry group analysis deals with applications of the continuous transformation groups to the differential equations (linear or nonlinear ordinary and partial differential equations). However, in the case of integro-differential equations (IDEs), there is no general method for investigating symmetry groups of these equations based on the solutions of their determining equations. The main difficulty in applying Lie's infinitesimal techniques to these systems is their nonlocality, and the approach used in the classical group theory cannot be applied for the investigation of symmetry groups of IDEs. In the literature, there are a few studies dealing with symmetry group analysis of IDEs. Özer (2003a,b) and Özer (1999) studied the symmetry groups of the one and two-dimensional nonlocal elasticity equations, and obtained a classification with respect to the free term and the kernel function of the integral equation. He also investigated

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symmetry group properties of the Benney equations in the Lagrangian variables and obtained reduced forms of the system (Özer, 2004a). Meleshko (1988) studied Lie point symmetries of the one-dimensional visco-elasticity equations. Ibragimov et al. (2002) investigated symmetry groups of the Benney equations based on the canonical multiplication law (Lie-Bäcklund or generalized symmetry approach) and they presented an infinite hierarchy of conservation laws corresponding exactly to the generalized symmetries (Olver, 1986). Krasnoslobodtsev (1989) obtained symmetry groups of the Benney equations by using the method of moments, which is based on the fact that the basic system of the integro-differential equation, which includes a distribution function and the moments of this function, is reduced to an infinite system of differential equations by using the moments. In addition, Roberts (1985) and Zawistowski (2001) studied symmetry groups of the one-dimensional Vlasov–Maxwell equations using the infinitesimal criterion of IDEs. Taranov (1976) studied symmetry groups of the one-dimensional Vlasov-Maxwell equations by using a different method in which the first step is to find the symmetries of the equation that depends on finite number of variables. Then the symmetries of the system of IDEs are obtained by approaching the number of variables to infinity. Bobylev (1993); Chetverikov and Kudryavtsev (1995) are other authors who studied symmetry groups of problems with IDEs.

The Benney equations, which are derived from the two-dimensional and time-dependent motion of an inviscid homogeneous fluid in a gravitational field by assuming the depth of the fluid to be small compared to the horizontal wavelengths considered, are expressed as

$$\begin{aligned} &\frac{\partial u(x,y,t)}{\partial t}+u(x,y,t)\frac{\partial u(x,y,t)}{\partial x}-\frac{\partial u(x,y,t)}{\partial y}\int_0^y\frac{\partial u(x,\tau,t)}{\partial x}\,\mathrm{d}\tau+\frac{\partial h(x,t)}{\partial x}=0,\\ &\frac{\partial h(x,t)}{\partial t}+\frac{\partial}{\partial x}\int_0^h u(x,\tau,t)\,\mathrm{d}\tau=0. \end{aligned}\tag{1.1}$$

where $y = 0$ is the rigid bottom, $y = h(x,t)$ is the free surface, and $u(x,y,t)$ is the horizontal velocity component. In this case, if the horizontal velocity component $u$ is independent of height $h$, and equation system (1.1) reduces to the equation system in the classical water theory corresponding to the case of irrotational motion. The corresponding wave motion is determined by the coupled one-dimensional nonlinear shallow-water system:

$$\begin{aligned} &\frac{\partial h(x,t)}{\partial t}+u(x,t)\frac{\partial h(x,t)}{\partial x}+h(x,t)\frac{\partial u(x,t)}{\partial x}=0,\\ &\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}+\frac{\partial h(x,t)}{\partial x}=0. \end{aligned}\tag{1.2}$$

Benney (1973) investigated the initial value problem for $u = u(x,y,0)$ and $h = h(x,0)$ for the functions $u(x,y,t)$ and $h(x,t)$. This form of the equations is called *fully nonlinear long-wave equations*. He introduced an infinite number of conservation laws for Eqs. (1.1) as an unclosed set of equations for the moments:

$$\frac{\partial A_n}{\partial t}+\frac{\partial A_{n+1}}{\partial x}+nA_{n-1}\frac{\partial A_0}{\partial x}=0,\tag{1.3}$$

where

$$A_n=\int_0^h u^n(x,y,t)\,\mathrm{d}y,\quad n=0,1,2,\ldots\tag{1.4}$$

Kupershmidt and Manin (1977) showed that Eq. (1.3) is a Hamiltonian system that can be written using the Poisson bracket defined as