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Probabilistic Engineering Mechanics 20 (2005) 45–56

PROBABILISTIC ENGINEERING MECHANICS

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An extension of the Kazakov relationship for non-Gaussian random variables and its use in the non-linear stochastic dynamics

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Received 24 March 2003; revised 11 June 2004; accepted 24 June 2004

Abstract

A generalization for non-Gaussian random variables of the well-known Kazakov relationship is reported in this work. If applied to the stochastic linearization of non-linear systems under non-Gaussian excitations, this relationship allows us to define the significance of the linearized stiffness coefficient. It is the sum of that one known in the literature (the mean of the tangent stiffness) and of terms taking into account the non-Gaussianity of the response. Moreover, the relationship here given is used for finding alternative formulae between the moments and the quasi-moments. Lastly, it is used in the framework of the moment equation approach, coupled with a quasi-moment neglect closure, for solving non-linear systems under Gaussian or non-Gaussian forces. In this way an iterative procedure based on the solution of a linear differential equation system, in which the values of the response mean and variance are those of the precedent iteration, is originated. It reveals a good level of accuracy and a fast convergence.

Keywords: Stochastic non-linear systems; Non-Gaussian processes; Stochastic linearization; Moment equation approach; Quasi-moment neglect closure

1. Introduction

When a non-linear dynamical system excited by a random Gaussian excitation is solved by means of the equivalent stochastic linearization, the response is approximated to be a Gaussian process. In the evaluation of the approximated Gaussian response X, a fundamental role is played by the evaluation of the quantity $E[f(X)X]$. Kazakov [1] showed that, in case of zero-mean process X, this quantity is given as $E[\mathrm{d}f(X)/\mathrm{d}X]E[X^2]$. This expression, establishing a direct relationship between $E[f(X)X]$ and the mean square value of X, makes the evaluation of the approximated response of the non-linear system simpler. But this expression is important from a theoretical point of view, too. In fact it establishes that the stiffness coefficient in the linearized system is equal to $E[\mathrm{d}f(X)/\mathrm{d}X]$, that is the mean of the tangent stiffness of the non-linear system. Moreover, this relationship can be used for determining the relationships between the moments of higher order of a Gaussian random variable X and the corresponding mean square value.

Later, Atalik and Utku [2] used this fundamental relationship for applying the stochastic equivalent linearization to non-linear structural systems. Spanos [3] was the first researcher to use it in the field of the non-stationary stochastic equivalent linearization. Falsone [4] extended this relationship to the case of non-zero-mean Gaussian variables, for which the knowledge of the mean of X is required, besides its variance. Even this last relationship can be advantageously used in the stochastic equivalent linearization and in the determination of the relationships between the moments of higher order and the first two ones when the Gaussian variable has not zero-mean.

In the present work, the extensions of the previous relationships are obtained in the case in which the response is approximated as a non-Gaussian process. In particular, it is assumed that its probability density is given by the truncated type-A Gram–Charlier expansion [5]. It is important to note that Beaman and Hedrick [6] used the Gram–Charlier expansion in the field of the stochastic linearization for the first time. In particular, they treated

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0266-8920/$ - see front matter
doi:10.1016/j.probengmech.2004.06.001

the case of non-linear system excited by Gaussian white noises. But here the obtained relationships are advantageously applied when the stochastic linearization is applied for the random analysis of non-linear systems excited by non-Gaussian excitations. In fact, in these cases the response is not Gaussian even if the system is linearised. Moreover, the relationship here obtained allows us to apply the moment equation approach coupled with a quasi-moment neglect closure in a simple form. As it is known, these closure techniques give better results than the equivalent stochastic linearization [7], taking into account the effective non-Gaussianity of the response of non-linear systems excited by random excitations. Finally, this generalisation of the Kazakov relationship lets us find very useful expressions between the moments of any order of random non-Gaussian variables and the corresponding Hermite moments or quasi-moments.

It is important to note that the aim of this work is not the introduction of new methods for solving non-linear random problems. But it is the use of the founded extended relationship to some existing techniques for evidencing some theoretical and computational features.

In the following sections, firstly, the fundamental steps in order to extend the Kazakov relationship to non-Gaussian variable cases are given. Secondly, its use in the stochastic linearization of non-linear systems excited by Gaussian and non-Gaussian excitations is shown. Then the application of the relationship here obtained to the moment equation approach coupled with a quasi-moment neglect closure is presented. Lastly, some numerical examples are reported with the aim of showing the computational usefulness of these relationships.

For clarity's sake, the case of a non-linear first-order system characterised by one state variable is treated. Then multi-state variable systems are considered, showing as the corresponding formulations are a simple generalization of the preceding ones, as it should be.

2. Extension of the Kazakov relationship. The case of a single non-Gaussian variable

First the case of a single non-Gaussian random variable will be treated. Then the formulation will be extended to the case of a vector of non-Gaussian variables.

Let us assume that the probability density function of a non-Gaussian variable X is expressed as follows

$$p_X(x) = \left(1 + \sum_{j=3}^{\infty} \frac{1}{j!} C_j H_j(\bar{x})\right) p_X^0(x) \tag{1}$$

where C_j is the jth order Hermite moment of X, that is $C_j = E[H_j(\bar{X})]$; $H_j(\cdot)$ is the jth order Hermite polynomial of $(\cdot)$; $\bar{X} = (X - E[X])/\sigma_X$ is the standardized random variable; $\bar{x} = (x - E[X])/\sigma_X$ is the standardized variable; p_X^0 is the Gaussian probability density function having the same mean and variance of p_X. The expression given in Eq. (1) is known as type-A Gram–Charlier expansion of p_X [5].

From the definition of the stochastic mean it is possible to write

$$E\left[\frac{\mathrm{d}f(X)}{\mathrm{d}X}\right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}f(x)}{\mathrm{d}x} p_X(x)\mathrm{d}x = -\int_{-\infty}^{\infty} \frac{\mathrm{d}p_X(x)}{\mathrm{d}x} f(x)\mathrm{d}x \tag{2}$$

where the integration by parts and the assumption $|f(X)| < c\exp(X^\alpha)$ (with $\alpha < 2$, for some arbitrary c and any X) have been taken into account. By considering Eq. (1) and some properties of the Hermite polynomials (given in Appendix B), Eq. (2) gives

$$E\left[\frac{\mathrm{d}f(X)}{\mathrm{d}X}\right] = \int_{-\infty}^{\infty} \left(\sum_{j=3}^{\infty} \frac{1}{(j-1)!} \frac{1}{\sigma_X} C_j H_{j-1}(\bar{x})\right) f(x) p_X^0(x)\mathrm{d}x + \int_{-\infty}^{\infty} \frac{1}{\sigma_X} \bar{x} f(x) p_X(x)\mathrm{d}x \tag{3}$$

that, after simple algebra, allows us to write:

$$E[f(X)X] = E\left[\frac{\mathrm{d}f(X)}{\mathrm{d}X}\right]\sigma_X^2 + E[f(X)]E[X] + \sum_{j=3}^{\infty} \frac{1}{(j-1)!} C_j \sigma_X E^0[H_{j-1}(\bar{X}) f(X)] \tag{4}$$

$E^0[\cdot]$ being the stochastic mean of $(\cdot)$ made with reference to the Gaussian probability density function p_X^0. It is not difficult to see that the relationship given in Eq. (4) is a generalisation of the Kazakov relationship to a non-Gaussian non-zero mean random variable. In fact, if X is Gaussian the last term of Eq. (4) vanishes, the Hermite moments of greater order than 2 being 0 for a Gaussian variable. Moreover, if the variable is zero-mean, the second term of the second member of Eq. (4) vanishes, too. In this way, Eq. (4) coincides with the Kazakov relationship.

In the following it will be seen that, in the field of non-Gaussian random processes, the relationship in Eq. (4) has the same relevance as the Kazakov relationship has for Gaussian processes. This happens from both a computational and, above all, a theoretical point of view.

For example, let us consider a non-linear system excited by a non-Gaussian excitation, which is a zero-mean delta-correlated process $\bar{W}(t)$, governed by the following equation:

$$\dot{X}(t) = f(X(t)) + \bar{W}(t) \tag{5}$$

When the stochastic linearization is applied, the corresponding linear system has the form

$$\dot{X}(t) = k_e X(t) + a_e + \bar{W}(t) \tag{6}$$